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A multiple-scales analysis of total internal reflection at a fluid–solid interface

Z M Rogoff† and R H Tew‡

Department of Theoretical Mechanics, University of Nottingham, Nottingham NG7 2RD, UK

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Abstract. This paper concerns the total internal reflection of an elastic wavefield at the boundary between a compressible fluid and an adjacent elastic solid. The system is forced by a prescribed high-frequency wave incident upon the interface from the fluid and emphasis is placed on the construction of the local acoustic response near the points where the shear or longitudinal components of the total elastic field propagate tangentially to the boundary.

A methodology for this construction, which is based on multiple-scales techniques, is presented and detailed calculations are carried out for both flat and curved interfaces. These analyses not only yield local amplitude balances and governing solutions but also furnish certain diffraction coefficients which might be of importance in other global scattering problems involving coupled fluid–solid configurations.

1. Introduction

In this paper, we give a detailed account of one very important aspect of the high-frequency diffraction of a two-dimensional acoustic wave when incident upon a (possibly curved) common interface between a compressible fluid and an elastic solid, namely total internal reflection. To be more specific, we are interested in the wave structure local to the point (or points) on the boundary where an incident ray, such as is emitted from a non-uniform acoustic line source, meets the boundary and generates transmitted longitudinal and shear rays such that one or other propagates (at least locally) parallel to the boundary.

To illustrate this in simple terms, consider the two-dimensional problem of an isotropic acoustic line-source situated many wavelengths away from a flat fluid–solid interface. Decomposing the radiation from the source into its constituent expansion fan of rays, a typical incoming ray will partially reflect and partially transmit on impinging the boundary, with the transmitted elastic ray field comprising both longitudinal (P) and shear (S) rays. Ignoring the relative amplitudes of the fields along these rays and assuming that the speed of sound in the fluid is less than that of shear wave propagation in the solid, the angle of ray transmission into the solid (measured from the normal pointing into the solid) increases with the angle of incidence (again measured from the normal, this time into the fluid). There are clearly two critical angles of incidence, θ_P and θ_S (with $\theta_P < \theta_S$), for which the transmitted P- and S-rays, respectively, propagate parallel to (and therefore along) the boundary. As the angle of incidence increases past θ_α ($\alpha = P, S$), the α-type transmitted

† Present address: St Petersburg Branch, Steklov Mathematical Institute, Fontanka 27, St Petersburg D-11, 191011, Russia.

‡ Author to whom correspondence should be addressed. E-mail address: richard.tew@nottingham.ac.uk

ray becomes complex and an associated evanescent field results (see Chapman *et al* (1998) for a discussion of this). However, the transmitted surface rays that are excited precisely at the critical boundary points (henceforth referred to as points of total internal reflection) continue to propagate along the boundary decaying according to an inverse three-halves power law with propagated distance.

Of course, whilst we are guaranteed that the boundary conditions are satisfied at the critical point itself, the critically transmitted surface ray generated at this point will not satisfy them by itself at other more general points on the boundary and extra diffracted ray fields must be introduced to account for this. One way of thinking about this is that the mechanical interaction of each critically transmitted ray with the boundary induces further plane waves in *both* fluid and solid. These waves are called 'head' or 'lateral' waves (Brekhovskikh and Godin 1992) and inherit the same algebraic decay law with distance as the parent critical surface ray (which is sometimes referred to as a 'surface skimming bulk wave' by researchers in non-destructive testing).

In this particular case, the wave speeds are such that the acoustic head waves—there are two, one for each critically transmitted elastic ray—propagate without exponential decay into the fluid. The S-type elastic head wave induced in the solid by the critically transmitted P-ray also propagates without exponential decay into the solid though the P-type head wave generated by the critical S-type transmitted ray is evanescent.

One crucial feature to note is that none of these surface fields decay *exponentially* with distance in the direction of propagation along the boundary. Hence, once they are excited they are capable of propagating for significant distances with measurable amplitudes. They are therefore potentially very useful indeed in non-destructive testing evaluations, such as those obtained by using the acoustic microscope (Briggs 1992, 1995).

The above description for head-wave propagation is mainly relevant to the case of a two-dimensional, flat interface with obvious extensions to the three-dimensional case. If the interface is curved, and for definiteness is taken to be concave on the elastic side, then the so-called 'whispering gallery' modes are excited (Babič and Buldyrev 1991, Ludwig 1975). These can be thought of as 'interior creeping fields', where by the term 'creeping field' we mean the surface ray and associated diffracted field generated by exterior tangential ray incidence upon a convex boundary (Keller and Lewis, 1995).

Be they flat or curved interfaces, general ray-type constructions for the critically transmitted (i.e. totally internally reflected) surface rays and associated diffracted fields exist (Keller and Lewis 1995) for simple boundary conditions, which permit extension to other more general circumstances.

Whichever situation we consider, the ray solution alone will never give us a complete theory since certain amplitude (or 'diffraction') coefficients will be missing. These are governed by the excitation process in the neighbourhood of the critical point of total internal reflection, where we know ray theory to be inapplicable. This difficulty is usually overcome by posing an appropriate 'canonical' full wave problem (rather than a ray approximation to one) in this neighbourhood, solving it and then matching its far field to the ray solution to the actual problem that we are considering. As we shall note presently, such a canonical problem might very well be the line-source problem referred to previously—this is amenable to an exact integral Fourier transform solution and the head-wave contributions then arise via branch point singularities of the integrand (Brekhovskikh and Godin 1992, Tew 1992a). However, this solution cannot be used universally on all two-dimensional fluid–solid head-wave calculations and the following remarks may help us to see why.

Our first observation is that any canonical problem must take curvature effects into account, either in the wavefronts of the incoming field (as in the analysis of Tew 1992a) or else in the boundary. If not, then the issue is to consider critical plane-wave incidence upon a flat fluid–solid boundary, the solution to which is easy to obtain (Brekhovskikh and Godin 1990) but which has shortcomings as far as we are concerned.

For example, whichever elastic wave it is that is being totally internally reflected, it will be a full, plane wave propagating parallel to the boundary in the solid. The associated ray picture for this particular wave is then a family of straight lines running parallel to the boundary and occupying the whole of the elastic half-space. Hence, in this formulation the critically transmitted field at any point in the solid away from the boundary will be due to a unique ray passing through that point which *never intersects the boundary*. This ray cannot then be identified as a 'transmitted ray' as such and so this solution can never match into that for more realistic circumstances.

Given that we have now identified curvature as a necessary feature of the inner canonical problem, we now assume that on the local inner scale the wavefront curvature either greatly exceeds, or is significantly less than, the boundary curvature.

If we allow curvature in the wavefronts and take a planar boundary, then the appropriate canonical problem is that of the isotropic line-source referred to previously. The argument is that even if the actual source is non-uniform and produces a non-isotropic expansion fan of incident rays, those rays that impinge the boundary near the point of total internal reflection will come from a very narrow pencil of rays emitted from the source—so narrow that there is no leading-order angular amplitude variation from one included ray to the next. They therefore all appear to be identical as far as the inner diffraction problem is concerned and so the source may as well be assumed to be uniform for these purposes.

The canonical diffraction problem for the situation with curvature in the boundary is critical plane-wave incidence on an interface with a gradual modulation in curvature. Since this will involve analysing the *source-free* Helmholtz equation in curvilinear coordinates, there is no reasonable expectation that we should be able to recover the solution for this case from that of the previous canonical case.

Indeed, the second situation does not admit an exact solution for arbitrary curvature and so we must devise an asymptotic method to solve this inner diffraction problem from which the diffraction coefficients required for the global scattering problem—which will not be discussed in detail here though this analysis will form the basis of a subsequent paper—can be read off.

We have developed such a methodology based on the multiple-scales analysis of a related, but much simpler, problem (Tew and Ockendon 1992). It can also be applied to the curved wavefront-flat boundary case and there is actually a significant advantage in doing so rather than on relying on the far-field asymptotics of the exact integral transform solution that is available in that case (Tew 1992a). This is because it gives the appropriate asymptotic balances near the point of total internal reflection, and the corresponding solutions, in a direct and natural way and this is information which is difficult (though not impossible) to extract from the integral solution. At the very least, it gives us an opportunity to check our results against those from the known exact solution before we apply it to the second case where no other solution exists.

We now continue the main body of the paper with detailed analyses of the wave solution local to points of total internal reflection, first when the wavefronts are curved and the boundary is flat and then for flat wavefronts impinging a curved boundary.

2. Total internal reflection-curved wavefronts, flat boundary

2.1. Formulation of the problem

In this case, the curvature of the wavefronts greatly exceeds that of the boundary in the vicinity of the critical point and, as has been observed earlier, the source can then be taken to be uniform.

We adopt a Cartesian coordinate system such that the undisturbed fluid–solid interface lies along $y = 0$ with the fluid occupying the half-space $y > 0$ and the solid $y < 0$. If the acoustic source is located at the point $(0, h)$ then the critical boundary points of total internal reflection are $(x_\alpha, 0) = (h \cot\theta_\alpha, 0)$, $\alpha = P, S$, where the angles $\theta_\alpha = \cos^{-1}(c_0/c_\alpha)$ are as depicted in figure 1 and where c_0 is the acoustic wavespeed and c_α is the speed of propagation of the α-type elastic wave ($c_0 < c_S < c_P$).

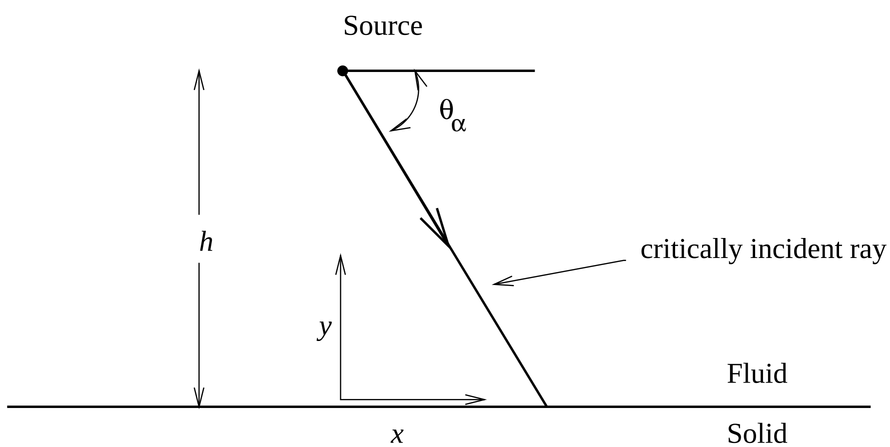

Figure 1. The scattering geometry and the critical angle of incidence.

To describe the wave structure, we universally suppress a time-harmonic factor $e^{-i\omega t}$ and introduce a velocity potential for the fluid and elastic displacement potentials for the solid. More specifically, we express the *total* fluid velocity potential as a superposition of the Green function to account for the uniform source at the point $(0, h)$, the Green function for the corresponding image source at the point $(0, -h)$ and a 'scattered' potential ϕ. This decomposition simplifies some of the boundary conditions and is done purely for convenience. For the elastic displacements, we shall work with the scalar potentials ψ and χ such that the displacement vector is given by $\boldsymbol{u}(x, y) = \boldsymbol{\nabla}\psi(x, y) + \boldsymbol{\nabla} \times (\chi(x, y)\boldsymbol{k})$.

In order to identify the correct inner problem, we must scale the dependent and independent variables appropriately. Under the high-frequency assumption (which can be interpreted as meaning $k_\alpha = \omega/c_\alpha$ ($\alpha = 0, P, S$) are all large), it is appropriate to scale

$$x = x_\alpha + k_0^{-1}\hat{x} \qquad y = k_0^{-1}\hat{y} \qquad |\hat{x}|, |\hat{y}| = \mathcal{O}(1) \tag{2.1}$$

and

$$(\phi, \psi, \chi) = \frac{-\varepsilon c_P e^{i\pi/4}}{\sqrt{\pi} k_0 \sin\theta_\alpha} \exp\left(\frac{-ik_0 h}{\sin\theta_\alpha}\right)(\hat{\phi}_\alpha, \hat{\psi}_\alpha, \hat{\chi}_\alpha). \tag{2.2}$$

The parameter ε in (2.2) is defined to be $(c_0\rho_F)/(c_P\rho_S)$, where ρ_F, ρ_S are the densities of the fluid and solid, respectively. We take care to note that whilst other aspects of the global scattering problem, such as the launching of a leaky Rayleigh wave (Tew 1995), works on the assumption that $0 < \varepsilon \ll 1$, this particular diffraction analysis does not need to do so (though our theory here is trivial to adjust if ε is small). We choose to leave the ε dependence explicit so that cross-reference can easily be made to these other analyses, if need be.

With these scalings, the boundary value problem to be solved is given by

$$\left(\hat{\nabla}^2 + 1\right)\hat{\phi}_\alpha = 0 \qquad \hat{y} > 0 \tag{2.3}$$

$$\left(\hat{\nabla}^2 + \cos^2\theta_P\right)\hat{\psi}_\alpha = 0 \qquad \hat{y} < 0 \tag{2.4}$$

$$\left(\hat{\nabla}^2 + \cos^2\theta_S\right)\hat{\chi}_\alpha = 0 \qquad \hat{y} < 0 \tag{2.5}$$

with boundary conditions

$$2\frac{\partial^2\hat{\psi}_\alpha}{\partial\hat{x}\partial\hat{y}} + \frac{\partial^2\hat{\chi}_\alpha}{\partial\hat{y}^2} - \frac{\partial^2\hat{\chi}_\alpha}{\partial\hat{x}^2} = 0 \tag{2.6}$$

$$c_P^2\left(\frac{\partial^2\hat{\psi}_\alpha}{\partial\hat{x}^2} + \frac{\partial^2\hat{\psi}_\alpha}{\partial\hat{y}^2}\right) - 2c_S^2\frac{\partial^2\hat{\psi}_\alpha}{\partial\hat{x}^2} - 2c_S^2\frac{\partial^2\hat{\chi}_\alpha}{\partial\hat{x}\partial\hat{y}} + \frac{\mathrm{i}\varepsilon c_0 c_P}{\omega}\hat{\phi}_\alpha = \delta_\alpha^{1/2}\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_\alpha + \mathrm{i}\delta_\alpha\hat{x}^2} \tag{2.7}$$

$$\mathrm{i}\omega\left(\frac{\partial\hat{\psi}_\alpha}{\partial\hat{y}} - \frac{\partial\hat{\chi}_\alpha}{\partial\hat{x}}\right) + \frac{\partial\hat{\phi}_\alpha}{\partial\hat{y}} = 0 \tag{2.8}$$

all to be evaluated on $\hat{y} = 0$. In addition, the solutions for $\hat{\phi}_\alpha$, $\hat{\psi}_\alpha$ and $\hat{\chi}_\alpha$ must all exhibit appropriate behaviour at infinity.

The first two of the boundary conditions guarantee continuity of surface traction across the interface and the third represents continuity of normal component of velocity.

The forcing term on the right-hand side of (2.7) requires explanation; first, the parameter δ_α is a small, dimensionless quality such that

$$0 < \delta_\alpha = \frac{\sin^3\theta_\alpha}{2k_0 h} \ll 1. \tag{2.9}$$

Second, the forcing term arises from the presence of the two Green function terms in the *total* velocity potential. It turns out that in the neighbourhood being examined the arguments of both terms are uniformly large and we are then justified in replacing these terms by their leading-order asymptotic expansions. The scalings (2.1) then permit a further approximation to the phase and amplitude of the forcing term and (2.7) is the upshot.

2.2. *Multiple-scales analysis*

Following the method of Tew and Ockendon (1992), we adopt a multiple-scales approach using the two sets of slow variables

$$(X, Y) = \delta_\alpha^{1/2}(\hat{x}, \hat{y}) \qquad \text{and} \qquad (\bar{X}, \bar{Y}) = \delta_\alpha^{1/4}(\hat{x}, \hat{y}). \tag{2.10}$$

We also assume expansions for the potentials in the form

$$\hat{\phi}_\alpha = \delta_\alpha^{1/2}\hat{\phi}_2 + \delta_\alpha^{3/4}\hat{\phi}_3 + \delta_\alpha\hat{\phi}_4 + \delta_\alpha{}^{5/4}\hat{\phi}_5 + \cdots \tag{2.11}$$

where the expansions for $\hat{\psi}_\alpha$ and $\hat{\chi}_\alpha$ follow similarly. We deliberately start the expansion at $\mathcal{O}(\delta_\alpha^{1/2})$ since the coefficient functions $\hat{\phi}_0$ and $\hat{\phi}_1$, at $\mathcal{O}(1)$ and $\mathcal{O}(\delta_\alpha^{1/4})$, respectively, are identically zero. In this account, we shall consider $\varepsilon = \mathcal{O}(1)$. The corrections to account for the case $\varepsilon \ll 1$, as is relevant for light fluid loading but which is of no particular significance here, are trivially done by stating in advance the relative sizes of ε and $\delta_\alpha^{1/4}$ and inserting the ε-dependent terms at the correct order in the expansion.

Analysis at $\mathcal{O}\big(\delta_\alpha{}^{1/2}\big)$. The $\mathcal{O}\big(\delta_\alpha{}^{1/2}\big)$ boundary value problem is to solve the appropriate Helmholtz equation for each of $\hat{\phi}_2$, $\hat{\psi}_2$, $\hat{\chi}_2$ with boundary conditions

$$2\frac{\partial^2\hat{\psi}_2}{\partial\hat{x}\partial\hat{y}}+\frac{\partial^2\hat{\chi}_2}{\partial\hat{y}^2}-\frac{\partial^2\hat{\chi}_2}{\partial\hat{x}^2}=0 \tag{2.12}$$

$$c_P^2\hat{\nabla}^2\psi_2-2c_S^2\frac{\partial^2\hat{\psi}_2}{\partial\hat{x}^2}-2c_S^2\frac{\partial^2\hat{\chi}_2}{\partial\hat{x}\partial\hat{y}}+\frac{\varepsilon\mathrm{i}c_0c_P}{\omega}\hat{\phi}_2=\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_\alpha+\mathrm{i}X^2} \tag{2.13}$$

$$\mathrm{i}\omega\left(\frac{\partial\hat{\psi}_2}{\partial\hat{y}}-\frac{\partial\hat{\chi}_2}{\partial\hat{x}}\right)+\frac{\partial\hat{\phi}_2}{\partial\hat{y}}=0 \tag{2.14}$$

all on $\hat{y}=0$. Motivated by the boundary condition (2.13) we consider solutions in plane-wave form. When the P-wave is generated tangential to the boundary (i.e. $\alpha=P$), it is appropriate to seek solutions of the form

$$\hat{\phi}_2(X,Y;\hat{x},\hat{y})=A_2^{(P)}(X,Y)\,\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_P+\mathrm{i}\hat{y}\sin\theta_P}\qquad \hat{y}>0 \tag{2.15}$$

$$\hat{\psi}_2(\bar{X},\bar{Y};X,Y;\hat{x})=B_2^{(P)}(\bar{X},\bar{Y};X,Y)\,\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_P}\qquad \hat{y}<0 \tag{2.16}$$

$$\hat{\chi}_2(X,Y;\hat{x},\hat{y})=C_2^{(P)}(X,Y)\,\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_P-\mathrm{i}\hat{y}(\cos^2\theta_S-\cos^2\theta_P)^{1/2}}\qquad \hat{y}<0 \tag{2.17}$$

whilst those for which S-wave is tangential ($\alpha=S$) are given by

$$\hat{\phi}_2(X,Y;\hat{x},\hat{y})=A_2^{(S)}(X,Y)\,\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_S+\mathrm{i}\hat{y}\sin\theta_S}\qquad \hat{y}>0 \tag{2.18}$$

$$\hat{\psi}_2(X,Y;\hat{x},\hat{y})=C_2^{(S)}(X,Y)\,\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_S+\hat{y}(\cos^2\theta_S-\cos^2\theta_P)^{1/2}}\qquad \hat{y}<0 \tag{2.19}$$

$$\hat{\chi}_2(\bar{X},\bar{Y};X,Y;\hat{x})=B_2^{(S)}(\bar{X},\bar{Y};X,Y)\,\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_S}\qquad \hat{y}<0. \tag{2.20}$$

Notice that we have placed an extra dependence on the overbarred variables on the fields which are being totally internally reflected. As will become apparent, the structure of these fields possesses a rapidly varying amplitude which necessarily depends on these extra variables whereas the other amplitudes do not. We emphasize that we have denoted by $B_2^{(\alpha)}$ the amplitude of the tangential field in both instances.

Substitution of the plane-wave forms (2.15)–(2.17) and, separately, (2.18)–(2.20) into the boundary conditions (2.12)–(2.14) generates the ‘reduced boundary conditions’, given by

$\alpha=P$:

$$A_2^{(P)}(X,0)=0 \tag{2.21}$$

$$B_2^{(P)}(\bar{X},0;X,0)=\frac{\mathrm{e}^{\mathrm{i}X^2}}{\cos^2\theta_P\big(2c_S^2-c_P^2\big)} \tag{2.22}$$

$$C_2^{(P)}(X,0)=0 \tag{2.23}$$

$\alpha=S$:

$$A_2^{(S)}(X,0)=\frac{\omega\,\mathrm{e}^{\mathrm{i}X^2}}{c_0c_P\Big(\big(c_S^2-c_0^2\big)^{1/2}/\big(c_P^2-c_S^2\big)^{1/2}+\mathrm{i}\varepsilon\Big)} \tag{2.24}$$

$$B_2^{(S)}(\bar{X},0;X,0)=-\frac{2\mathrm{i}\big(c_P^2-c_S^2\big)^{1/2}\mathrm{e}^{\mathrm{i}X^2}}{c_Pc_0^2\Big(1+\mathrm{i}\varepsilon\big(c_P^2-c_S^2\big)^{1/2}/\big(c_S^2-c_0^2\big)^{1/2}\Big)} \tag{2.25}$$

$$C_2^{(S)}(X,0)=\frac{\mathrm{e}^{\mathrm{i}X^2}}{c_0^2\Big(1+\mathrm{i}\varepsilon\big(c_P^2-c_S^2\big)^{1/2}/\big(c_S^2-c_0^2\big)^{1/2}\Big)}. \tag{2.26}$$

Analysis at $\mathcal{O}(\delta_\alpha^{3/4})$. Given the results so far, the field equation for $\hat{\phi}_3$ at this order is given by

$$(\hat{\nabla}^2 + 1)\hat{\phi}_3 + 2\left(\frac{\partial^2}{\partial\hat{x}\partial\bar{X}} + \frac{\partial^2}{\partial\hat{y}\partial\bar{Y}}\right)\hat{\phi}_2 = 0 \qquad \hat{y} > 0 \tag{2.27}$$

with equations for $\hat{\psi}_2$ and $\hat{\chi}_2$ following similarly. The associated boundary conditions are

$$2\frac{\partial^2\hat{\psi}_3}{\partial\hat{x}\partial\hat{y}} + 2\left(\frac{\partial^2}{\partial\hat{x}\partial\bar{Y}} + \frac{\partial^2}{\partial\hat{y}\partial\bar{X}}\right)\hat{\psi}_2 + \frac{\partial^2\hat{\chi}_3}{\partial\hat{y}^2} + 2\frac{\partial^2\hat{\chi}_2}{\partial\hat{y}\partial\bar{Y}} - \frac{\partial^2\hat{\chi}_3}{\partial\hat{x}^2} - 2\frac{\partial^2\hat{\chi}_2}{\partial\hat{x}\partial\bar{X}} = 0 \tag{2.28}$$

$$c_P^2\hat{\nabla}^2\hat{\psi}_3 + 2c_P^2\left(\frac{\partial^2}{\partial\hat{x}\partial\bar{X}} + \frac{\partial^2}{\partial\hat{y}\partial\bar{Y}}\right)\hat{\psi}_2 - 2c_S^2\frac{\partial^2\hat{\psi}_3}{\partial\hat{x}^2} - 4c_S^2\frac{\partial^2\hat{\psi}_2}{\partial\hat{x}\partial\bar{X}}$$
$$- 2c_S^2\frac{\partial^2\hat{\chi}_3}{\partial\hat{x}\partial\hat{y}} - 2c_S^2\left(\frac{\partial^2}{\partial\hat{x}\partial\bar{Y}} + \frac{\partial^2}{\partial\hat{y}\partial\bar{X}}\right)\hat{\chi}_2 + \frac{\varepsilon \mathrm{i} c_0 c_P}{\omega}\hat{\phi}_3 = 0 \tag{2.29}$$

$$\mathrm{i}\omega\left(\frac{\partial\hat{\psi}_3}{\partial\hat{y}} + \frac{\partial\hat{\psi}_2}{\partial\bar{Y}} - \frac{\partial\hat{\chi}_3}{\partial\hat{x}} - \frac{\partial\hat{\chi}_2}{\partial\bar{X}}\right) + \frac{\partial\hat{\phi}_3}{\partial\hat{y}} + \frac{\partial\hat{\phi}_2}{\partial\bar{Y}} = 0 \tag{2.30}$$

all on $\hat{y} = \bar{Y} = 0$.

In order to suppress secular growth in higher-order terms in the expansions for $\hat{\psi}_\alpha, \hat{\phi}_\alpha$ and $\hat{\chi}_\alpha$, we must take $B_2^{(\alpha)} = B_2^{(\alpha)}(\bar{Y}; X, Y)$ and

$$A_2^{(\alpha)} = F_2^{(\alpha)}(\eta_\alpha) \qquad C_2^{(\alpha)} = G_2^{(\alpha)}(\zeta_\alpha) \tag{2.31}$$

where

$$\eta_\alpha = X - \cot\theta_\alpha Y \tag{2.32}$$

$$\zeta_P = X + Y\cos\theta_P/\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2} \tag{2.33}$$

$$\zeta_S = X - \mathrm{i}Y\cos\theta_S/\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}. \tag{2.34}$$

From (2.31), (2.21), (2.23), (2.24) and (2.26) we obtain

$$F_2^{(P)}(\eta_P) = G_2^{(P)}(\zeta_P) = 0 \tag{2.35}$$

$$\left.\begin{aligned} F_2^{(S)}(\eta_S) &= \frac{\omega\,\mathrm{e}^{\mathrm{i}\eta_S^2}}{c_0 c_P\left(\left(c_S^2 - c_0^2\right)^{1/2}/\left(c_P^2 - c_S^2\right)^{1/2} + \mathrm{i}\varepsilon\right)} \\ G_2^{(S)}(\zeta_S) &= \frac{\mathrm{e}^{\mathrm{i}\zeta_S^2}}{c_0^2\left(1 + \mathrm{i}\varepsilon\left(c_P^2 - c_S^2\right)^{1/2}/\left(c_S^2 - c_0^2\right)^{1/2}\right)} \end{aligned}\right. \tag{2.36}$$

leaving only $B_2^{(\alpha)}$ to be determined to close the leading-order solution. This requires us to examine the problem for $\hat{\phi}_3$, $\hat{\psi}_3$ and $\hat{\chi}_3$, the boundary conditions for which now follow as

$$2\frac{\partial^2\hat{\psi}_3}{\partial\hat{x}\partial\hat{y}} + \frac{\partial^2\hat{\chi}_3}{\partial\hat{y}^2} - \frac{\partial^2\hat{\chi}_3}{\partial\hat{x}^2} = \begin{cases} -2\mathrm{i}\cos\theta_P\dfrac{\partial B_2}{\partial\bar{Y}}^{(P)}\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_P} \\ 0 \end{cases} \tag{2.37}$$

$$c_P^2\hat{\nabla}^2\hat{\psi}_3 - 2c_S^2\frac{\partial^2\hat{\psi}_3}{\partial\hat{x}^2} - 2c_S^2\frac{\partial^2\hat{\chi}_3}{\partial\hat{x}\partial\hat{y}} + \frac{\varepsilon\mathrm{i}c_0c_P}{\omega}\hat{\phi}_3 = \begin{cases} 0 \\ 2\mathrm{i}c_0c_S\dfrac{\partial B_2}{\partial\bar{Y}}^{(S)}\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_S} \end{cases} \tag{2.38}$$

$$\mathrm{i}\omega\left(\frac{\partial\hat{\psi}_3}{\partial\hat{y}} - \frac{\partial\hat{\chi}_3}{\partial\hat{x}}\right) + \frac{\partial\hat{\phi}_3}{\partial\hat{y}} = \begin{cases} -\mathrm{i}\omega\dfrac{\partial B_2}{\partial\bar{Y}}^{(P)}\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_P} \\ 0 \end{cases} \tag{2.39}$$

all on $\hat{y} = \bar{Y} = 0$. The upper/lower forcing terms correspond to the tangential P/S-waves, respectively.

Motivated by the forcing terms in these boundary conditions we also have plane-wave solutions for $\hat{\phi}_3$, $\hat{\psi}_3$ and $\hat{\chi}_3$, given by (2.15)–(2.17) or (2.18)–(2.20) with the subscripts 2 replaced by 3. When substituted into the above boundary conditions, we obtain the 'reduced boundary data'

$\alpha = P$:

$$A_3^{(P)}(X,0) = \frac{\omega c_P^2}{\sin\theta_P\left(2c_S^2 - c_P^2\right)} \frac{\partial B_2}{\partial \bar{Y}}^{(P)}(\bar{Y}=0; X, 0) \tag{2.40}$$

$$B_3^{(P)}(\bar{X},0;X,0) = -\frac{\mathrm{i}c_0\left(4c_S^3\left(c_S^2 - c_P^2\right)^{1/2} + c_P^4\varepsilon/\sin\theta_P\right)}{c_P\left(2c_S^2 - c_P^2\right)^2} \frac{\partial B_2}{\partial \bar{Y}}^{(P)}(\bar{Y}=0; X, 0) \tag{2.41}$$

$$C_3^{(P)}(X,0) = -\frac{2\mathrm{i}c_P c_S^2}{c_0\left(2c_S^2 - c_P^2\right)} \frac{\partial B_2}{\partial \bar{Y}}^{(P)}(\bar{Y}=0; X, 0) \tag{2.42}$$

$\alpha = S$:

$$A_3^{(S)}(X,0) = \frac{2\mathrm{i}\omega c_S}{c_P\left(\left(c_S^2 - c_0^2\right)^{1/2}/\left(c_P^2 - c_S^2\right)^{1/2} + \mathrm{i}\varepsilon\right)} \frac{\partial B_2}{\partial \bar{Y}}^{(S)}(\bar{Y}=0; X, 0) \tag{2.43}$$

$$B_3^{(S)}(\bar{X},0;X,0) = \frac{4c_S\left(c_P^2 - c_S^2\right)^{1/2}}{c_P c_0\left(1 + \mathrm{i}\varepsilon\left(c_P^2 - c_S^2\right)^{1/2}/\left(c_S^2 - c_0^2\right)^{1/2}\right)} \frac{\partial B_2}{\partial \bar{Y}}^{(S)}(\bar{Y}=0; X, 0) \tag{2.44}$$

$$C_3^{(S)}(X,0) = \frac{2\mathrm{i}c_S}{c_0\left(1 + \mathrm{i}\varepsilon\left(c_P^2 - c_S^2\right)^{1/2}/\left(c_S^2 - c_0^2\right)^{1/2}\right)} \frac{\partial B_2}{\partial \bar{Y}}^{(S)}(\bar{Y}=0; X, 0). \tag{2.45}$$

Further information is obtained by now considering higher order terms in the expansions.

Analysis at $\mathcal{O}(\delta_\alpha)$ Substitution of the plane-wave forms for $\hat{\phi}_2$, $\hat{\psi}_2$, $\hat{\chi}_2$, $\hat{\phi}_3$, $\hat{\psi}_3$ and $\hat{\chi}_3$ into the governing equations at $\mathcal{O}(\delta_\alpha)$ yields the field equation

$$\left(\hat{\nabla}^2 + 1\right)\hat{\phi}_4 + 2\left(\frac{\partial^2}{\partial\hat{x}\partial\bar{X}} + \frac{\partial^2}{\partial\hat{y}\partial\bar{Y}}\right)\hat{\phi}_3 + \left(2\frac{\partial^2}{\partial\hat{x}\partial X} + 2\frac{\partial^2}{\partial\hat{y}\partial Y} + \bar{\nabla}^2\right)\hat{\phi}_2 = 0 \tag{2.46}$$

with similar equations following for $\hat{\psi}_4$ and $\hat{\chi}_4$.

For the tangential elastic fields we find that the usual appeal to secularity arguments leads us to

$$2\mathrm{i}\cos\theta_\alpha \frac{\partial B_3}{\partial \bar{X}}^{(\alpha)} + 2\mathrm{i}\cos\theta_\alpha \frac{\partial B_2}{\partial X}^{(\alpha)} + \frac{\partial^2 B_2}{\partial \bar{Y}^2}^{(\alpha)} = 0 \qquad \bar{Y} < 0. \tag{2.47}$$

Since $B_2^{(\alpha)}$ is independent of $\bar{X}$, possible integration of (2.47) with respect to $\bar{X}$ demands we must set

$$2\mathrm{i}\cos\theta_\alpha \frac{\partial B_2}{\partial X}^{(\alpha)} + \frac{\partial^2 B_2}{\partial \bar{Y}^2}^{(\alpha)} = 0 \qquad \bar{Y} < 0 \tag{2.48}$$

in order to avoid secular growth of $B_3^{(\alpha)}$ in $\bar{X}$. In fact, this also implies that $B_3^{(\alpha)}$ is independent of $\bar{X}$.

At this stage of the calculation it is helpful to isolate the boundary value problem for $B_2^{(\alpha)}(\bar{Y}; X, Y)$ as follows:

$$2\mathrm{i}\cos\theta_\alpha \frac{\partial B_2}{\partial X}^{(\alpha)} + \frac{\partial^2 B_2}{\partial \bar{Y}^2}^{(\alpha)} = 0 \qquad \bar{Y} < 0 \tag{2.49}$$

$$B_2^{(P)}(0; X, 0) = \frac{\mathrm{e}^{\mathrm{i}X^2}}{\cos^2\theta_P\left(2c_S^2 - c_P^2\right)} \tag{2.50}$$

$$B_2^{(S)}(0; X, 0) = -\frac{2\mathrm{i}\left(c_P^2 - c_S^2\right)^{1/2}\mathrm{e}^{\mathrm{i}X^2}}{c_P c_0^2\left(1 + \mathrm{i}\varepsilon\left(c_P^2 - c_S^2\right)^{1/2}/\left(c_S^2 - c_0^2\right)^{1/2}\right)}. \tag{2.51}$$

It is apparent that the problem for $B_2^{(\alpha)}(\bar{Y}; X, Y)$ is still not closed since we do not have any information on how $B_2^{(\alpha)}$ depends on Y. To rectify this, we must examine further the boundary value problem at $\mathcal{O}(\delta_\alpha)$ and $\mathcal{O}\left(\delta_\alpha^{5/4}\right)$.

Substituting the plane-wave forms for $\hat{\phi}_2, \hat{\psi}_2, \hat{\chi}_2, \hat{\phi}_3, \hat{\psi}_3$ and $\hat{\chi}_3$ into the boundary conditions arising at $\mathcal{O}(\delta_\alpha)$ we obtain boundary conditions for $\hat{\phi}_4$, $\hat{\psi}_4$ and $\hat{\chi}_4$ with forcing terms possessing the spatial factor $\mathrm{e}^{\mathrm{i}\hat{x}\cos\theta_\alpha}$. This implies that the plane-wave forms for $\hat{\phi}_4$, $\hat{\psi}_4$ and $\hat{\chi}_4$ are also of the form as those contained in (2.15)–(2.20) with the subscript 2 now replaced by 4. Substituting these acoustic plane-wave forms into the governing acoustic equation at $\mathcal{O}\left(\delta_\alpha^{5/4}\right)$ and invoking secularity arguments, we obtain

$$A_3^{(\alpha)} = F_3^{(\alpha)}(\eta_\alpha) \tag{2.52}$$

with η_α given by (2.32). It then follows from (2.40) that

$$F_3^{(P)}(X) = \frac{\omega c_P^2}{\sin\theta_P\left(2c_S^2 - c_P^2\right)} \frac{\partial B_2}{\partial \bar{Y}}^{(P)}(\bar{Y} = 0; X, 0) \tag{2.53}$$

and from (2.43) that

$$F_3^{(S)}(X) = \frac{2\mathrm{i}\omega c_S}{c_P\left(\left(c_S^2 - c_0^2\right)^{1/2}/\left(c_P^2 - c_S^2\right)^{1/2} + \mathrm{i}\varepsilon\right)} \frac{\partial B_2}{\partial \bar{Y}}^{(S)}(\bar{Y} = 0; X, 0). \tag{2.54}$$

Thus, once we construct $B_2^{(\alpha)}$, we can close the leading-order solution everywhere as well as the second-order solution in the fluid. To establish the form of $B_2^{(\alpha)}$, we note that the appropriate elastic field equations at $\mathcal{O}\left(\delta_\alpha^{5/4}\right)$ yield

$$2\mathrm{i}\cos\theta_\alpha \frac{\partial B_4}{\partial \bar{X}}^{(\alpha)} + 2\mathrm{i}\cos\theta_\alpha \frac{\partial B_3}{\partial X}^{(\alpha)} + \frac{\partial^2 B_3}{\partial \bar{Y}^2}^{(\alpha)} + 2\frac{\partial^2 B_2^{(\alpha)}}{\partial Y \partial \bar{Y}} = 0 \qquad \bar{Y}, Y < 0. \tag{2.55}$$

Since both $B_2^{(\alpha)}$ and $B_3^{(\alpha)}$ are independent of $\bar{X}$, equation (2.55) demands we set

$$2\mathrm{i}\cos\theta_\alpha \frac{\partial B_3}{\partial X}^{(\alpha)} + \frac{\partial^2 B_3}{\partial \bar{Y}^2}^{(\alpha)} + 2\frac{\partial^2 B_2^{(\alpha)}}{\partial Y \partial \bar{Y}} = 0 \qquad \bar{Y}, Y < 0 \tag{2.56}$$

in order to avoid secular growth of $B_4^{(\alpha)}$ with respect to $\bar{X}$.

We are now able to determine $B_2^{(\alpha)}$ and we begin by defining the Fourier transform

$$\tilde{f}(\bar{Y}; \xi, Y) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi X} f(\bar{Y}; X, Y)\,\mathrm{d}X \tag{2.57}$$

with inverse

$$f(\bar{Y}; X, Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi X} \tilde{f}(\bar{Y}; \xi, Y)\,\mathrm{d}\xi \tag{2.58}$$

for a general function f. Applying the transform (2.57) to (2.49) gives us

$$\frac{\partial^2 \tilde{B}_2^{(\alpha)}}{\partial \bar{Y}^2} - 2\xi \cos\theta_\alpha \tilde{B}_2^{(\alpha)} = 0 \qquad \bar{Y} < 0. \tag{2.59}$$

Solving this using the outgoing wave/extinction condition as $\bar{Y} \to -\infty$ gives

$$\tilde{B}_2^{(\alpha)}(\bar{Y}; \xi, Y) = b(\xi, Y)\, \mathrm{e}^{\sqrt{2\xi\cos\theta_\alpha}\bar{Y}} \qquad \bar{Y} < 0 \tag{2.60}$$

where we define the branch of the square root by $\xi^{1/2} = +\sqrt{\xi}$ for $\xi > 0$ and $\xi^{1/2} = -\mathrm{i}\sqrt{|\xi|}$ for $\xi < 0$. Consideration of the Fourier transform of the equation (2.56) yields

$$\frac{\partial^2 \tilde{B}_3^{(\alpha)}}{\partial \bar{Y}^2} - 2\xi \cos\theta_\alpha \tilde{B}_3^{(\alpha)} = -2\sqrt{2\xi\cos\theta_\alpha}\frac{\partial b}{\partial Y}\mathrm{e}^{\sqrt{2\xi\cos\theta_\alpha}\bar{Y}} \tag{2.61}$$

whereupon, by secularity arguments, we must set

$$\frac{\partial b}{\partial Y}(\xi, Y) = 0 \tag{2.62}$$

which implies that $b = b(\xi)$. Therefore $\tilde{B}_2^{(\alpha)} = \tilde{B}_2^{(\alpha)}(\bar{Y}, \xi) = b(\xi)\,\mathrm{e}^{\sqrt{2\xi\cos\theta_\alpha}\bar{Y}}$. By taking the Fourier transform of the boundary conditions (2.50) and (2.51) in turn we may calculate $b(\xi)$ and then use the inversion formula (2.58) to establish that

$$B_2^{(\alpha)}(\bar{Y}, X) = E_\alpha \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}X\xi + \sqrt{2\xi\cos\theta_\alpha}\bar{Y} - \mathrm{i}\xi^2/4}\,\mathrm{d}\xi \qquad \bar{Y} < 0 \tag{2.63}$$

where

$$E_P = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{2\sqrt{\pi}\cos^2\theta_P\left(2c_S^2 - c_P^2\right)} \qquad E_S = -\frac{\mathrm{i}\left(c_P^2 - c_S^2\right)^{1/2}\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{\pi}c_P c_0^2\left(1 + \mathrm{i}\varepsilon\left(c_P^2 - c_S^2\right)^{1/2}/\left(c_S^2 - c_0^2\right)^{1/2}\right)}. \tag{2.64}$$

This finally allows us to calculate $F_3^{(\alpha)}$ from (2.53) and (2.54) in the forms

$$F_3^{(P)}(X) = \frac{2^{3/4}\omega c_P^2\mathrm{e}^{-3\pi\mathrm{i}/8}}{\sin\theta_P\cos^{3/2}\theta_P\left(2c_S^2 - c_P^2\right)^2} D_{1/2}\left(\sqrt{2}\,X\mathrm{e}^{3\pi\mathrm{i}/4}\right)\mathrm{e}^{\mathrm{i}X^2/2} \tag{2.65}$$

$$F_3^{(S)}(X) = \frac{2^{11/4}c_S\omega\left(c_P^2 - c_S^2\right)\sqrt{\cos\theta_S}\,\mathrm{e}^{-3\pi\mathrm{i}/8}}{c_P^2c_0^2\left(c_S^2 - c_0^2\right)^{1/2}\left(1 + \mathrm{i}\varepsilon\left(c_P^2 - c_S^2\right)^{1/2}/\left(c_S^2 - c_0^2\right)^{1/2}\right)^2} D_{1/2}\left(\sqrt{2}\,X\mathrm{e}^{3\pi\mathrm{i}/4}\right)\mathrm{e}^{\mathrm{i}X^2/2} \tag{2.66}$$

where we have used the identity

$$\int_{-\infty}^{\infty} \xi^{1/2}\mathrm{e}^{\mathrm{i}X\xi - \mathrm{i}\xi^2/4}\,\mathrm{d}\xi = 2^{5/4}\sqrt{\pi}\,\mathrm{e}^{-5\pi\mathrm{i}/8}\mathrm{e}^{\mathrm{i}X^2/2}D_{1/2}\left(\sqrt{2}\,X\mathrm{e}^{3\pi\mathrm{i}/4}\right) \tag{2.67}$$

to evaluate the solution in terms of the parabolic cylinder function of order one-half $D_{1/2}(z)$.

2.3. The acoustic response

We are now in a position to use our results to state the acoustic response in the vicinity of the point of total internal reflection of the α-type elastic wave. In fact, if we use (2.2) to reinstate the scaling prefactors, we obtain

$$\phi \sim V_\alpha\varepsilon\delta_\alpha^{1/2}\mathrm{e}^{\mathrm{i}k_0h/\sin\theta_\alpha + \mathrm{i}\hat{x}\cos\theta_\alpha + \mathrm{i}\hat{y}\sin\theta_\alpha + \mathrm{i}\eta_\alpha^2} + W_\alpha\varepsilon\delta_\alpha^{3/4}\mathrm{e}^{\mathrm{i}k_0h/\sin\theta_\alpha + \mathrm{i}\hat{x}\cos\theta_\alpha + \mathrm{i}\hat{y}\sin\theta_\alpha + \mathrm{i}\eta_\alpha^2/2}D_{1/2}\left(\sqrt{2}\,\eta_\alpha\mathrm{e}^{3\pi\mathrm{i}/4}\right) \qquad \hat{y} > 0 \tag{2.68}$$

where

$$V_P = 0 \tag{2.69}$$

$$V_S = \frac{e^{3\pi i/4}}{\sin\theta_S\sqrt{\pi}\left(\left(c_S^2 - c_0^2\right)^{1/2}/\left(c_P^2 - c_S^2\right)^{1/2} + i\varepsilon\right)} \tag{2.70}$$

$$W_P = \frac{2^{3/4}c_P^4 e^{3\pi i/8}}{\sqrt{\pi}\sin^2\theta_P\cos^{1/2}\theta_P\left(2c_S^2 - c_P^2\right)^2} \tag{2.71}$$

$$W_S = \frac{2^{11/4}c_S^2\sqrt{\cos\theta_S}\,e^{3\pi i/8}}{c_P c_0\sqrt{\pi}\left(\left(c_S^2 - c_0^2\right)^{1/2}/\left(c_P^2 - c_S^2\right)^{1/2} + i\varepsilon\right)^2}. \tag{2.72}$$

Expanding this solution for large $|\eta_\alpha|$ we obtain, using standard asymptotic expansions for the parabolic cylinder functions,

$$\phi \sim V_\alpha\varepsilon\delta_\alpha^{1/2}e^{ik_0h/\sin\theta_\alpha + i\hat{x}\cos\theta_\alpha + i\hat{y}\sin\theta_\alpha + i\eta_\alpha^2} + H_\alpha\varepsilon\delta_\alpha^{3/4}e^{ik_0h/\sin\theta_\alpha + i\hat{x}\cos\theta_\alpha + i\hat{y}\sin\theta_\alpha + i\eta_\alpha^2}\left(|\eta_\alpha|^{1/2} + \frac{i}{16|\eta_\alpha|^{3/2}}\right) \qquad \eta_\alpha < 0 \tag{2.73}$$

while for $\eta_\alpha > 0$

$$\begin{aligned}\phi \sim {} & V_\alpha\varepsilon\delta_\alpha^{1/2}e^{ik_0h/\sin\theta_\alpha + i\hat{x}\cos\theta_\alpha + i\hat{y}\sin\theta_\alpha + i\eta_\alpha^2} \\ & + iH_\alpha\varepsilon\delta_\alpha^{3/4}e^{ik_0h/\sin\theta_\alpha + i\hat{x}\cos\theta_\alpha + i\hat{y}\sin\theta_\alpha + i\eta_\alpha^2}\left(\eta_\alpha^{1/2} + \frac{i}{16\eta_\alpha^{3/2}}\right) \\ & + \frac{iH_\alpha\varepsilon}{2^{3/2}}\delta_\alpha^{3/4}\frac{e^{ik_0h/\sin\theta_\alpha + i\hat{x}\cos\theta_\alpha + i\hat{y}\sin\theta_\alpha}}{\eta_\alpha^{3/2}}\end{aligned} \tag{2.74}$$

where

$$H_P = \frac{2c_P^4 e^{\pi i/4}}{\sqrt{\pi}\sin^2\theta_P\cos^{1/2}\theta_P\left(2c_S^2 - c_P^2\right)^2} \tag{2.75}$$

$$H_S = \frac{8c_S^2\sqrt{\cos\theta_S}\,e^{\pi i/4}}{c_P c_0\sqrt{\pi}\left(\left(c_S^2 - c_0^2\right)^{1/2}/\left(c_P^2 - c_S^2\right)^{1/2} + i\varepsilon\right)^2}. \tag{2.76}$$

The interpretation of these results is that (2.68) is the local acoustic response near the point of total internal reflection and its asymptotic limit, given by (2.73) and (2.74), must match into the outer ray field. The three terms common to (2.73) and (2.74) must match into the specularly reflected field and the final term in (2.74) must match the acoustic head wave.

Independent ray calculations, details of which are presented in the appendix, confirm that there is indeed a precise match between inner and outer reflected fields. If we introduce a set of polar coordinates (R, θ) based on the image source point $(0, -h)$, then another independent ray calculation, also presented in the appendix, shows that the outer form of the acoustic head wave induced by the total internal reflection of the α-type elastic wave is

$$\phi^{(\alpha H)} \sim Q_\alpha\frac{e^{ik_0R\cos(\theta_\alpha - \theta)}}{(k_0R\sin(\theta_\alpha - \theta))^{3/2}} \qquad \theta < \theta_\alpha \tag{2.77}$$

where the Q_α are unknown diffraction coefficients, the successful determination of which is one of the principal aims of this calculation.

Matching between (2.74) and (2.77) gives us that

$$Q_P = \frac{c_P^4 e^{3\pi i/4}}{\surd(2\pi \sin\theta_P \cos\theta_P)\left(2c_S^2 - c_P^2\right)^2} \tag{2.78}$$

$$Q_S = \frac{2^{3/2} c_S^2 \surd\left(\cos\theta_S \sin^3\theta_S\right) e^{3\pi i/4}}{\sqrt{\pi}\, c_P c_0 \left(\left(\left(c_S^2 - c_0^2\right)^{1/2} / \left(c_P^2 - c_S^2\right)^{1/2}\right) + i\varepsilon\right)^2} \tag{2.79}$$

in precise agreement with the results of Tew (1992a). We have therefore succeeded in describing the process of the total internal reflection of an elastic wave at a flat fluid–solid boundary and have confirmed the accuracy of the results via comparison with an exact analysis. This now gives confidence in the methodology and we apply it to the more general case of a curved interface, for which no such exact solution exists.

3. Total internal reflection—planar wavefronts, curved boundary

3.1. Formulation of the problem

Consider now a time-harmonic plane wave propagating through the fluid towards the solid which now has a curved boundary. Without any loss of generality, we take the incoming field to be $\phi^{\text{inc}} = e^{ik_0 x - i\omega t}$ and we assume that the boundary appears convex from the fluid.

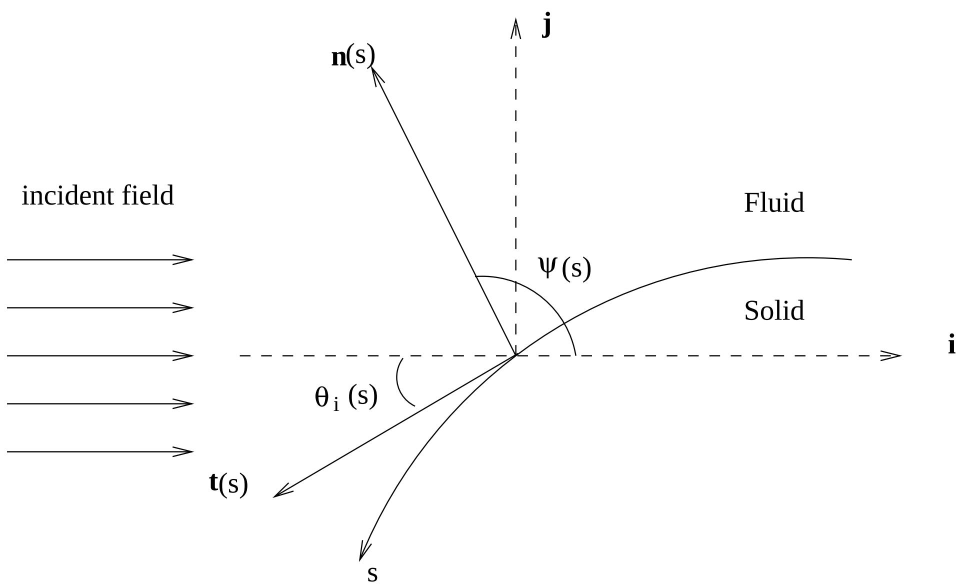

Figure 2. The scattering geometry for a curved fluid–solid interface.

The geometry and definitions of some useful angles and coordinates are shown in figure 2. In particular, we parametrize the boundary in terms of arc length s (which increases in an anticlockwise sense) in the form

$$x = x_0(s) \qquad y = y_0(s) \tag{3.1}$$

from which the unit tangent $\boldsymbol{t}(s)$ follows as $\boldsymbol{t}(s) = \boldsymbol{x}_0'(s)$ and the Senet–Frenet formulae (Struik 1988) can be used to establish relationships between $\boldsymbol{t}$ and the unit normal $\boldsymbol{n}(s)$ in terms of s and normal distance n.

It is no longer possible to simplify the full boundary value problem by subtracting out direct and image source terms and so we proceed by isolating just the incident field. Doing so means that the global problem is to solve three Helmholtz equations—akin to (2.3)–(2.5) though with full wavenumbers reinstated and which must be considered in terms of the boundary coordinates (s, n)—subject to the forced boundary conditions

$$2\left(-\kappa(s)\frac{\partial\psi}{\partial s + \partial^2\psi/\partial n\partial s}\right) - k_S^2\chi - 2\frac{\partial^2\chi}{\partial n^2} = 0 \tag{3.2}$$

$$-\left(k_S^2 - 2k_P^2\right)\psi + 2\frac{\partial^2\psi}{\partial n^2} + 2\left(-\kappa(s)\frac{\partial\chi}{\partial s} + \frac{\partial^2\psi}{\partial n\partial s}\right) + \frac{\mathrm{i}k_0k_S^2\varepsilon}{k_P\omega}\phi^{\mathrm{ref}} = -\frac{\mathrm{i}k_0k_S^2\varepsilon}{k_P\omega}\mathrm{e}^{\mathrm{i}k_0x_0(s)} \tag{3.3}$$

$$\frac{\partial\phi^{\mathrm{ref}}}{\partial n} + \mathrm{i}\omega\frac{\partial\psi}{\partial n} + \mathrm{i}\omega\frac{\partial\chi}{\partial s} = \mathrm{i}k_0\sin\theta_i(s)\,\mathrm{e}^{\mathrm{i}k_0x_0(s)} \tag{3.4}$$

all to be evaluated on $n = 0$. The parameter $\kappa(s)$ in (3.2) is the boundary curvature and the angle $\theta_i(s)$ arising in (3.4) is as depicted in figure 2. In this formulation, we have written the total acoustic potential as $\phi^{\mathrm{inc}} + \phi^{\mathrm{ref}}$ where the second term includes all diffraction effects, as well as specular reflections. Analogous decompositions are taken for the elastic potentials.

The incoming rays which will give rise to total internal reflection are those for which $\cos\theta_i(s) = c_0/c_\alpha$ (i.e. $\theta_i(s) = \theta_\alpha = \cos^{-1}(c_0/c_\alpha)$) and we suppose that this occurs at $s = s_\alpha$. The appropriate inner scalings are then

$$(s, n) = \left(s_\alpha + k_0^{-1}\hat{s}, k_0^{-1}\hat{n}\right) \qquad (\alpha = P, S) \tag{3.5}$$

which will lead to the normalized Holmholtz equations (2.3)–(2.5) which can further approximated by

$$\left[\left(\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + 1\right) + \frac{2\delta_\alpha}{\sin\theta_\alpha}\left(\frac{\partial}{\partial\hat{n}} - 2\hat{n}\frac{\partial^2}{\partial\hat{s}^2}\right) + \mathrm{O}\left(\delta_\alpha^2\right)\right]\hat{\phi}^{\mathrm{ref}}(\hat{s}, \hat{n}) = 0 \tag{3.6}$$

$$\left[\left(\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + \frac{c_0^2}{c_P^2}\right) + \frac{2\delta_\alpha}{\sin\theta_\alpha}\left(\frac{\partial}{\partial\hat{n}} - 2\hat{n}\frac{\partial^2}{\partial\hat{s}^2}\right) + \mathrm{O}\left(\delta_\alpha^2\right)\right]\hat{\psi}(\hat{s}, \hat{n}) = 0 \tag{3.7}$$

$$\left[\left(\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + \frac{c_0^2}{c_S^2}\right) + \frac{2\delta_\alpha}{\sin\theta_\alpha}\left(\frac{\partial}{\partial\hat{n}} - 2\hat{n}\frac{\partial^2}{\partial\hat{s}^2}\right) + \mathrm{O}\left(\delta_\alpha^2\right)\right]\hat{\chi}(\hat{s}, \hat{n}) = 0 \tag{3.8}$$

with boundary conditions

$$2\left(-\frac{2\delta_\alpha}{\sin\theta_\alpha}\frac{\partial\hat{\psi}}{\partial\hat{s}} + \frac{\partial^2\hat{\psi}}{\partial\hat{n}\partial\hat{s}}\right) - \frac{k_S^2}{{k_0}^2}\hat{\chi} - 2\frac{\partial^2\hat{\chi}}{\partial\hat{n}^2} \sim 0 \qquad n = 0 \tag{3.9}$$

$$-\frac{\left(k_S^2 - 2k_P^2\right)}{{k_0}^2}\hat{\psi} + 2\frac{\partial^2\hat{\psi}}{\partial\hat{n}^2} + 2\left(-\frac{2\delta_\alpha}{\sin\theta_\alpha}\frac{\partial\hat{\chi}}{\partial\hat{s}} + \frac{\partial^2\hat{\chi}}{\partial\hat{n}\partial\hat{s}}\right) + \frac{\mathrm{i}\varepsilon k_S^2}{k_Pk_0\omega}\hat{\phi}^{\mathrm{ref}}$$
$$\sim -\frac{\mathrm{i}\varepsilon k_S^2}{k_Pk_0\omega}\mathrm{e}^{\mathrm{i}k_0x_0(s_\alpha) - \mathrm{i}\hat{s}\cos\theta_\alpha + \mathrm{i}\delta_\alpha\hat{s}^2} \qquad n = 0 \tag{3.10}$$

$$\frac{\partial\hat{\phi}^{\mathrm{ref}}}{\partial\hat{n}} + \mathrm{i}\omega\left(\frac{\partial\hat{\psi}}{\partial\hat{n}} + \frac{\partial\hat{\chi}}{\partial\hat{s}}\right) \sim \mathrm{i}(\sin\theta_\alpha + 2\cot\theta_\alpha\delta_\alpha\hat{s})\,\mathrm{e}^{\mathrm{i}k_0x_0(s_\alpha) - \mathrm{i}\hat{s}\cos\theta_\alpha + \mathrm{i}\delta_\alpha\hat{s}^2} \qquad n = 0. \tag{3.11}$$

In this analysis the hats on the field variables denote that they are on an inner scale and δ_α is a small parameter given this time by

$$\delta_\alpha = \frac{\kappa(s_\alpha)\sin\theta_\alpha}{2k_0} \ll 1. \tag{3.12}$$

The parameter δ_α is taken to be small by virtue of the fact that the radius of curvature at s_α is much larger than any of the wavelengths in the problem. Whilst the combination of derivatives on the right-hand side of (3.9)–(3.11) are different from those in the previous case (cf equations (2.6)–(2.8)), there are similarities in that the forcing is of plane-wave type modulated in phase by a slowly varying quadratic dependence on $\hat{s}$. It is this that allows us to use the methods developed in the previous section, which is precisely what we do next.

3.2. Multiple-scales analysis

Proceeding in an identical fashion to section 2, we consider a multiple-scales approach with two sets of slow variables

$$(S, N) = \delta_\alpha^{1/2}(\hat{s}, \hat{n}) \qquad (\bar{S}, \bar{N}) = \delta_\alpha^{1/4}(\hat{s}, \hat{n}). \tag{3.13}$$

The expansion for the acoustic potential is taken to be

$$\hat{\phi}^{\text{ref}} = \hat{\phi}_0^{\text{ref}} + \delta_\alpha^{1/4}\hat{\phi}_1^{\text{ref}} + \delta_\alpha^{1/2}\hat{\phi}_2^{\text{ref}} + \delta_\alpha^{3/4}\hat{\phi}_3^{\text{ref}} + \delta_\alpha\hat{\phi}_4^{\text{ref}} + \cdots \tag{3.14}$$

with those for the elastic potentials being similar. Notice that this time the expansions begin at $\mathcal{O}(1)$—this is because we did not subtract out image source-type terms, as we did in section 2, and these $\mathcal{O}(1)$ terms account for leading-order reflections.

Again, we keep ε to leading order, under exactly the same conditions as in the previous section.

Analysis at $\mathcal{O}(1)$. Here the governing equations are given by

$$\left(\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + 1\right)\hat{\phi}_0^{\text{ref}} = 0 \qquad \hat{n} > 0 \tag{3.15}$$

$$\left(\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + \cos^2\theta_P\right)\hat{\psi}_0 = 0 \qquad \hat{n} < 0 \tag{3.16}$$

$$\left(\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + \cos^2\theta_S\right)\hat{\chi}_0 = 0 \qquad \hat{n} < 0 \tag{3.17}$$

with boundary conditions

$$2\frac{\partial^2\hat{\psi}_0}{\partial\hat{n}\partial\hat{s}} - \frac{k_S^2}{{k_0}^2}\hat{\chi}_0 - 2\frac{\partial^2\hat{\chi}_0}{\partial\hat{n}^2} = 0 \tag{3.18}$$

$$-\frac{\left(k_S^2 - 2k_P^2\right)}{{k_0}^2}\hat{\psi}_0 + 2\frac{\partial^2\hat{\psi}_0}{\partial\hat{n}^2} + 2\frac{\partial^2\hat{\chi}_0}{\partial\hat{n}\partial\hat{s}} + \frac{\mathrm{i}\varepsilon k_S^2}{k_P k_0\omega}\hat{\phi}_0^{\text{ref}} = -\frac{\mathrm{i}\varepsilon k_S^2}{k_P k_0\omega}\mathrm{e}^{\mathrm{i}k_0x_0(s_\alpha)-\mathrm{i}\hat{s}\cos\theta_\alpha+\mathrm{i}S^2} \tag{3.19}$$

$$\frac{\partial\hat{\phi}_0^{\text{ref}}}{\partial\hat{n}} + \mathrm{i}\omega\frac{\partial\hat{\psi}_0}{\partial\hat{n}} + \mathrm{i}\omega\frac{\partial\hat{\chi}_0}{\partial\hat{s}} = \mathrm{i}\sin\theta_\alpha\mathrm{e}^{\mathrm{i}k_0x_0(s_\alpha)-\mathrm{i}\hat{s}\cos\theta_\alpha+\mathrm{i}S^2} \tag{3.20}$$

all on $n = 0$. Whilst it has not been made explicit in the notation, $\hat{\psi}_0$ and $\hat{\chi}_0$ represent the leading-order elastic transmitted wavefields. In exactly the same way as before, we find

$\alpha = P$:

$$\hat{\phi}_0^{\text{ref}}(S, N; \hat{s}, \hat{n}) = A_0^{(P)}(S, N)\,\mathrm{e}^{-\mathrm{i}\hat{s}\cos\theta_P+\mathrm{i}\hat{n}\sin\theta_P} \qquad \hat{n} > 0 \tag{3.21}$$

$$\hat{\psi}_0(\bar{S}, \bar{N}; S, N; \hat{s}) = B_0^{(P)}(\bar{S}, \bar{N}; S, N)\,\mathrm{e}^{-\mathrm{i}\hat{s}\cos\theta_P} \qquad \hat{n} < 0 \tag{3.22}$$

$$\hat{\chi}_0(S, N; \hat{s}, \hat{n}) = C_0^{(P)}(S, N)\,\mathrm{e}^{-\mathrm{i}\hat{s}\cos\theta_P-\mathrm{i}\hat{n}(\cos^2\theta_S-\cos^2\theta_P)^{1/2}} \qquad \hat{n} < 0 \tag{3.23}$$

for the tangential P-field, and

$\alpha = S$:

$$\hat{\phi}_0^{\text{ref}}(S, N; \hat{s}, \hat{n}) = A_0^{(S)}(S, N)\, \mathrm{e}^{-\mathrm{i}\hat{s}\cos\theta_S + \mathrm{i}\hat{n}\sin\theta_S} \qquad \hat{n} > 0 \qquad (3.24)$$

$$\hat{\psi}_0(S, N; \hat{s}, \hat{n}) = C_0^{(S)}(S, N)\, \mathrm{e}^{-\mathrm{i}\hat{s}\cos\theta_S + \hat{n}(\cos^2\theta_S - \cos^2\theta_P)^{1/2}} \qquad \hat{n} < 0 \qquad (3.25)$$

$$\hat{\chi}_0(\bar{S}, \bar{N}; S, N; \hat{s}) = B_0^{(S)}(\bar{S}, \bar{N}; S, N)\, \mathrm{e}^{-\mathrm{i}\hat{s}\cos\theta_S} \qquad \hat{n} < 0 \qquad (3.26)$$

for the tangential S-field. Notice that these wavefields propagate along the boundary in the sense of *decreasing* $\hat{s}$, explaining the minus sign in the $\hat{s}$-dependent exponent. The associated boundary data is easily found to be

$\alpha = P$:

$$A_0^{(P)}(S, 0) = \mathrm{e}^{\mathrm{i}k_0 x_0(s_P) + \mathrm{i}S^2} \qquad (3.27)$$

$$B_0^{(P)}(\bar{S}, 0; S, 0) = \frac{2\mathrm{i}\varepsilon k_0 k_S^2}{\omega k_P\left(k_S^2 - 2k_P^2\right)} \mathrm{e}^{\mathrm{i}k_0 x_0(s_P) + \mathrm{i}S^2} \qquad (3.28)$$

$$C_0^{(P)}(S, 0) = 0 \qquad (3.29)$$

$\alpha = S$:

$$A_0^{(S)}(S, 0) = \frac{\sin\theta_S\cos\theta_P - \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}{\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}} \mathrm{e}^{\mathrm{i}k_0 x_0(s_S) + \mathrm{i}S^2} \qquad (3.30)$$

$$B_0^{(S)}(\bar{S}, 0; S, 0) = -\frac{4\varepsilon}{\omega} \frac{\tan\theta_S\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2} \mathrm{e}^{\mathrm{i}k_0 x_0(s_S) + \mathrm{i}S^2}}{\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}} \qquad (3.31)$$

$$C_0^{(S)}(S, 0) = -\frac{2\mathrm{i}\varepsilon}{\omega} \frac{\sin\theta_S \mathrm{e}^{\mathrm{i}k_0 x_0(s_S) + \mathrm{i}S^2}}{\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}. \qquad (3.32)$$

Analysis at $\mathcal{O}(\delta_\alpha^{1/4})$. We begin by substituting the leading-order plane-wave solutions just derived into the governing equations at $\mathcal{O}(\delta_\alpha^{1/4})$, given by

$$\left(\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + 1\right)\hat{\phi}_1^{\text{ref}} + 2\left(\frac{\partial^2}{\partial\hat{s}\partial\bar{S}} + \frac{\partial^2}{\partial\hat{n}\partial\bar{N}}\right)\hat{\phi}_0^{\text{ref}} = 0 \qquad \hat{n} > 0 \qquad (3.33)$$

with the other field equations being similar. Invoking secularity arguments we find we must set $\partial B_0^{(\alpha)}/\partial\bar{S} = 0$, i.e. $B_0^{(\alpha)} = B_0^{(\alpha)}(\bar{N}; S, N)$. The boundary conditions at $\mathcal{O}(\delta_\alpha^{1/4})$ are given by

$$2\frac{\partial^2\hat{\psi}_1}{\partial\hat{n}\partial\hat{s}} + 2\left(\frac{\partial^2}{\partial\hat{s}\partial\bar{N}} + \frac{\partial^2}{\partial\hat{n}\partial\bar{S}}\right)\hat{\psi}_0 - \frac{k_S^2}{{k_0}^2}\hat{\chi}_1 - 2\frac{\partial^2\hat{\chi}_1}{\partial\hat{n}^2} - 4\frac{\partial^2\hat{\chi}_0}{\partial\hat{n}\partial\bar{N}} = 0 \qquad (3.34)$$

$$-\frac{\left(k_S^2 - 2k_P^2\right)}{{k_0}^2}\hat{\psi}_1 + 2\frac{\partial^2\hat{\psi}_1}{\partial\hat{n}^2} + 4\frac{\partial^2\hat{\psi}_0}{\partial\hat{n}\partial\bar{N}} + 2\frac{\partial^2\hat{\chi}_1}{\partial\hat{n}\partial\hat{s}}$$
$$+2\left(\frac{\partial^2}{\partial\hat{s}\partial\bar{N}} + \frac{\partial^2}{\partial\hat{n}\partial\bar{S}}\right)\hat{\chi}_0 + \frac{\mathrm{i}\varepsilon k_S^2}{k_P k_0 \omega}\hat{\phi}_1^{\text{ref}} = 0 \qquad (3.35)$$

$$\frac{\partial\hat{\phi}_1}{\partial\hat{n}}^{\text{ref}} + \frac{\partial\hat{\phi}_0}{\partial\bar{N}}^{\text{ref}} + \mathrm{i}\omega\frac{\partial\hat{\psi}_1}{\partial\hat{n}} + \mathrm{i}\omega\frac{\partial\hat{\psi}_0}{\partial\bar{N}} + \mathrm{i}\omega\frac{\partial\hat{\chi}_1}{\partial\hat{s}} + \mathrm{i}\omega\frac{\partial\hat{\chi}_0}{\partial\bar{S}} = 0 \qquad (3.36)$$

all on $n = 0$.

Using the results obtained so far, the problem for $\hat{\phi}_1^{\text{ref}}$, $\hat{\psi}_1$ and $\hat{\chi}_1$ is to solve the appropriate Helmholtz equation—cross derivative terms in the lower-order terms like those in (3.33) being removed by secularity conditions (details to follow)—subject to the boundary conditions

$$2\frac{\partial^2\hat{\psi}_1}{\partial\hat{n}\partial\hat{s}} - \frac{k_S^2}{{k_0}^2}\hat{\chi}_1 - 2\frac{\partial^2\hat{\chi}_1}{\partial\hat{n}^2} = \begin{cases} 2\mathrm{i}\cos\theta_P \dfrac{\partial B_0}{\partial\bar{N}}^{(P)}\mathrm{e}^{-\mathrm{i}\hat{s}\cos\theta_P} \\ 0 \end{cases} \tag{3.37}$$

$$-\frac{\left(k_S^2 - 2k_P^2\right)}{{k_0}^2}\hat{\psi}_1 + 2\frac{\partial^2\hat{\psi}_1}{\partial\hat{n}^2} + 2\frac{\partial^2\hat{\chi}_1}{\partial\hat{n}\partial\hat{s}} + \frac{\mathrm{i}\varepsilon k_S^2}{k_P k_0\omega}\hat{\phi}_1^{\mathrm{ref}} = \begin{cases} 0 \\ 2\mathrm{i}\cos\theta_S \dfrac{\partial B_0}{\partial\bar{N}}^{(S)}\mathrm{e}^{-\mathrm{i}\hat{s}\cos\theta_S} \end{cases} \tag{3.38}$$

$$\frac{\partial\hat{\phi}_1}{\partial\hat{n}}^{\mathrm{ref}} + \mathrm{i}\omega\frac{\partial\hat{\psi}_1}{\partial\hat{n}} + \mathrm{i}\omega\frac{\partial\hat{\chi}_1}{\partial\hat{s}} = \begin{cases} -\mathrm{i}\omega\dfrac{\partial B_0}{\partial\bar{N}}^{(P)}\mathrm{e}^{-\mathrm{i}\hat{s}\cos\theta_P} \\ 0 \end{cases} \tag{3.39}$$

where the upper/lower forcings correspond to the tangential P/S-fields, respectively. This leads us to the anticipated plane-wave forms for $\hat{\phi}_1^{\mathrm{ref}}$, $\hat{\psi}_1$, $\hat{\chi}_1$ with boundary amplitudes given by

$\alpha = P$:

$$A_1^{(P)}(S,0) = -\frac{\omega\cos^2\theta_S}{\sin\theta_P\left(\cos^2\theta_S - 2\cos^2\theta_P\right)}\frac{\partial B_0}{\partial\bar{N}}^{(P)}(0;S,0) \tag{3.40}$$

$$B_1^{(P)}(\bar{S},0;S,0) = -\frac{\mathrm{i}\left[4\cos^2\theta_P\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2} + \frac{k_S^2\varepsilon}{k_P k_0}\frac{\cos^2\theta_S}{\sin\theta_P}\right]}{\left(\cos^2\theta_S - 2\cos^2\theta_P\right)^2}\frac{\partial B_0}{\partial\bar{N}}^{(P)}(0;S,0) \tag{3.41}$$

$$C_1^{(P)}(S,0) = \frac{2\mathrm{i}\cos\theta_P}{\cos^2\theta_S - 2\cos^2\theta_P}\frac{\partial B_0}{\partial\bar{N}}^{(P)}(0;S,0) \tag{3.42}$$

$\alpha = S$:

$$A_1^{(S)}(S,0) = \frac{2\mathrm{i}\omega\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}\cos\theta_P/\cos\theta_S}{\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}\frac{\partial B_0}{\partial\bar{N}}^{(S)}(0;S,0) \tag{3.43}$$

$$B_1^{(S)}(\bar{S},0;S,0) = \frac{\left(4\cos_P\theta_P\sin\theta_S/\cos^2\theta_S\right)\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}{\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}\frac{\partial B_0}{\partial\bar{N}}^{(S)}(0;S,0) \tag{3.44}$$

$$C_1^{(S)}(S,0) = \frac{2\mathrm{i}\cos\theta_P\tan\theta_S}{\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}\frac{\partial B_0}{\partial\bar{N}}^{(S)}(0;S,0). \tag{3.45}$$

Analysis at $\mathcal{O}\left(\delta_\alpha^{1/2}\right)$. It is here that we observe the first difference in the calculation from that performed in section 2 in that we now get field equations with non-constant coefficients. The appropriate governing equation at $\mathcal{O}\left(\delta_\alpha^{1/2}\right)$ for the acoustic response is

$$\left(\frac{\partial^2}{\partial\hat{s}^2} + \frac{\partial^2}{\partial\hat{n}^2} + 1\right)\hat{\phi}_2^{\mathrm{ref}} + 2\left(\frac{\partial^2}{\partial\hat{s}\partial\bar{S}} + \frac{\partial^2}{\partial\hat{n}\partial\bar{N}}\right)\hat{\phi}_1^{\mathrm{ref}} + \left(2\frac{\partial^2}{\partial\hat{s}\partial S} + 2\frac{\partial^2}{\partial\hat{n}\partial N} + \frac{\partial^2}{\partial\bar{S}^2} + \frac{\partial^2}{\partial\bar{N}^2} - \frac{4N}{\sin\theta_\alpha}\frac{\partial^2}{\partial\hat{s}^2}\right)\hat{\phi}_0^{\mathrm{ref}} = 0 \qquad \hat{n} > 0 \tag{3.46}$$

where the elastic equivalents are given similarly. Using the leading and second order acoustic plane-wave solutions obtained so far, along with secularity arguments, we obtain

$$\frac{\partial A_0}{\partial S}^{(\alpha)} - \tan\theta_\alpha \frac{\partial A_0}{\partial N}^{(\alpha)} + 2\mathrm{i}N\cot\theta_\alpha A_0^{(\alpha)} = 0 \qquad \hat{n} > 0. \tag{3.47}$$

This has solution

$$A_0^{(\alpha)}(S,N) = F_0^{(\alpha)}(\eta_\alpha)\,\mathrm{e}^{\mathrm{i}N^2\cot^2\theta_\alpha} \tag{3.48}$$

where $\eta_\alpha = S + N\cot\theta_\alpha$. Similarly, $C_0^{(\alpha)}$ is found to be

$$C_0^{(P)}(S,N) = G_0^{(P)}(\zeta_P)\exp\left[-\frac{\mathrm{i}N^2\cos^2\theta_P}{\sin\theta_P\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}\right] \tag{3.49}$$

$$C_0^{(S)}(S,N) = G_0^{(S)}(\zeta_S)\exp\left[-\frac{N^2\cos^2\theta_S}{\sin\theta_S\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}\right] \tag{3.50}$$

where $\zeta_P = S - N\cos\theta_P/\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}$ and $\zeta_S = S + \mathrm{i}N\cos\theta_S/\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}$. From (3.48) and (3.27), (3.29), (3.30) and (3.32) we obtain

$$F_0^{(\alpha)}(\eta_\alpha) = P_\alpha \mathrm{e}^{\mathrm{i}k_0x_0(s_\alpha)+\mathrm{i}\eta_\alpha^2} \tag{3.51}$$

where

$$P_P = 1 \qquad P_S = \frac{\sin\theta_S\cos\theta_P - \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}{\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}} \tag{3.52}$$

and

$$G_0^{(P)}(\zeta_P) = 0 \qquad G_0^{(S)}(\zeta_S) = \frac{-2\mathrm{i}\varepsilon\sin\theta_S \mathrm{e}^{\mathrm{i}k_0x_0(s_S)+\mathrm{i}\zeta_S^2}}{\omega\left(\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}\right)}. \tag{3.53}$$

For the tangential fields we find that secularity arguments at $\mathcal{O}\left(\delta_\alpha^{1/2}\right)$ imply

$$-2\mathrm{i}\cos\theta_\alpha \frac{\partial B_1}{\partial \bar{S}}^{(\alpha)} + \left(\frac{\partial^2 B_0}{\partial \bar{N}^2}^{(\alpha)} - 2\mathrm{i}\cos\theta_\alpha\frac{\partial B_0}{\partial S}^{(\alpha)} + \frac{4N\cos^2\theta_\alpha}{\sin\theta_\alpha}B_0^{(\alpha)}\right) = 0,\ N,\bar{N} < 0. \tag{3.54}$$

Since $B_0^{(\alpha)}(\bar{N};S,N)$ is independent of $\bar{S}$, this in turn implies that we must set

$$\frac{\partial^2 B_0}{\partial \bar{N}^2}^{(\alpha)} - 2\mathrm{i}\cos\theta_\alpha\frac{\partial B_0}{\partial S}^{(\alpha)} + \frac{4N\cos^2\theta_\alpha}{\sin\theta_\alpha}B_0^{(\alpha)} = 0 \qquad N,\bar{N} < 0 \tag{3.55}$$

to avoid secular growth of $B_1^{(\alpha)}$. We then observe that

$$B_0^{(\alpha)}(\bar{N};S,N) = \mathrm{e}^{-2\mathrm{i}SN\cot\theta_\alpha}\beta^{(\alpha)}(S,\bar{N}) \tag{3.56}$$

where

$$\frac{\partial^2\beta}{\partial\bar{N}^2}^{(\alpha)} - 2\mathrm{i}\cos\theta_\alpha\frac{\partial\beta}{\partial S}^{(\alpha)} = 0 \qquad \bar{N} < 0 \tag{3.57}$$

the transform of which has the solution

$$\tilde{\beta}^{(\alpha)}(\xi,\bar{N}) = b(\xi)\mathrm{e}^{\sqrt{2\xi\cos\theta_\alpha}\bar{N}} \qquad \bar{N} < 0 \tag{3.58}$$

where $\tilde{\beta}^{(\alpha)}$ is defined by (2.57). Boundary data for $\beta^{(\alpha)}(S, \bar{N})$ follows from (3.28) and (3.31) as

$$\beta^{(P)}(S,0) = \frac{2\mathrm{i}\varepsilon k_0 k_S^2}{\omega k_P\left(k_S^2 - 2k_P^2\right)}\mathrm{e}^{\mathrm{i}k_0 x_0(s_P)+\mathrm{i}S^2} \tag{3.59}$$

$$\beta^{(S)}(S,0) = -\frac{4\varepsilon}{\omega}\frac{\tan\theta_S\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}\mathrm{e}^{\mathrm{i}k_0 x_0(s_S)+\mathrm{i}S^2}}{\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}}. \tag{3.60}$$

This allows us to we determine $b(\xi)$, whereupon we obtain

$$B_0^{(\alpha)}(\bar{N}; SN) = E_\alpha \mathrm{e}^{\mathrm{i}k_0 x_0(s_\alpha) - 2\mathrm{i}SN\cot\theta_\alpha}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\xi S + \sqrt{2\cos\theta_\alpha\xi}\bar{N} - \mathrm{i}\xi^2/4}\,\mathrm{d}\xi \tag{3.61}$$

where the coefficients are given by

$$E_P = \frac{\mathrm{i}\varepsilon k_0 k_S^2 \mathrm{e}^{\mathrm{i}\pi/4}}{\omega\sqrt{\pi}k_P\left(k_S^2 - 2k_P^2\right)} \tag{3.62}$$

$$E_S = -\frac{2\varepsilon\tan\theta_S\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}\mathrm{e}^{\mathrm{i}\pi/4}}{\omega\sqrt{\pi}\left(\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}\right)}. \tag{3.63}$$

This now closes the leading-order solution but not that for the $\mathcal{O}\left(\delta_\alpha^{1/4}\right)$ correction term, in which the total internal reflection structure is embedded. For example, so far we know that these terms in the acoustic response have the same phase structure as (3.21) and (3.24) but the only thing we know about the amplitudes are their boundary values (3.40) and (3.43), which are now known to us via the above solution for $B_0^{(\alpha)}$.

Further information is obtained by looking at the $\mathcal{O}\left(\delta_\alpha^{3/4}\right)$ terms and the ubiquitous secularity arguments generates amplitudes of the form (3.48) with 0 the subscript replaced by 1. It then follows that

$$F_1^{(\alpha)}(\eta_\alpha) = Q_\alpha \mathrm{e}^{\mathrm{i}k_0 x_0(s_\alpha)+\mathrm{i}\eta_\alpha^2/2}D_{1/2}\left(\sqrt{2}\,\eta_\alpha\mathrm{e}^{-\mathrm{i}\pi/4}\right) \tag{3.64}$$

where

$$Q_P = -\frac{\varepsilon 2^{7/4}\cos^4\theta_S \mathrm{e}^{\mathrm{i}\pi/8}}{\sqrt{\cos\theta_P}\left(\cos^2\theta_S - 2\cos^2\theta_P\right)^2} \tag{3.65}$$

$$Q_S = -\frac{\varepsilon 2^{15/4}\cos\theta_P\tan\theta_S\left(\cos^2\theta_S - \cos^2\theta_P\right)\mathrm{e}^{\mathrm{i}\pi/8}}{\sqrt{\cos\theta_S}\left(\sin\theta_S\cos\theta_P + \mathrm{i}\varepsilon\left(\cos^2\theta_S - \cos^2\theta_P\right)^{1/2}\right)^2} \tag{3.66}$$

and $D_{1/2}(z)$ is the usual parabolic cylinder function.

3.3. The acoustic response

The expansion for the acoustic potential is determined to $\mathcal{O}\left(\delta_\alpha^{1/2}\right)$ to be

$$\hat{\phi}^{\mathrm{ref}} \sim \left\{P_\alpha \mathrm{e}^{\mathrm{i}\eta_\alpha^2} + \delta_\alpha^{1/4}Q_\alpha \mathrm{e}^{\mathrm{i}\eta_\alpha^2/2}D_{1/2}\left(\sqrt{2}\,\eta_\alpha \mathrm{e}^{-\mathrm{i}\pi/4}\right)\right\}\mathrm{e}^{\mathrm{i}k_0 x_0(s_\alpha)+\mathrm{i}N^2\cot^2\theta_\alpha - \mathrm{i}\hat{s}\cos\theta_\alpha + \mathrm{i}\hat{n}\sin\theta_\alpha}. \tag{3.67}$$

We are interested in the expansion of $\hat{\phi}$ for large $|\eta_\alpha|$. For $\eta_\alpha > 0$ the expansion is given by

$$\hat{\phi}^{\mathrm{ref}} \sim \left\{P_\alpha + Q_\alpha 2^{1/4}\mathrm{e}^{-\mathrm{i}\pi/8}\delta_\alpha^{1/4}\eta_\alpha^{1/2}\left(1 + \frac{\mathrm{i}}{16\eta_\alpha^2}\right)\right\} \times \mathrm{e}^{\mathrm{i}k_0 x_0(s_\alpha)+\mathrm{i}\eta_\alpha^2 + \mathrm{i}N^2\cot^2\theta_\alpha - \mathrm{i}\hat{s}\cos\theta_\alpha + \mathrm{i}\hat{n}\sin\theta_\alpha} \tag{3.68}$$

and for $\eta_\alpha < 0$ by

$$\hat{\phi}^{\text{ref}} \sim \left\{ P_\alpha + Q_\alpha 2^{1/4} \mathrm{e}^{3\mathrm{i}\pi/8} \delta_\alpha^{1/4} |\eta_\alpha|^{1/2} \left(1 + \frac{\mathrm{i}}{16\eta_\alpha^2}\right) \right\}$$
$$\times \mathrm{e}^{\mathrm{i}k_0 x_0(s_\alpha) + \mathrm{i}\eta_\alpha^2 + \mathrm{i}N^2 \cot^2\theta_\alpha - \mathrm{i}\hat{s}\cos\theta_\alpha + \mathrm{i}\hat{n}\sin\theta_\alpha}$$
$$+ \frac{Q_\alpha \delta_\alpha^{1/4} \mathrm{e}^{3\pi\mathrm{i}/8}}{2^{5/4}|\eta_\alpha|^{3/2}} \mathrm{e}^{\mathrm{i}k_0 x_0(s_\alpha) - \mathrm{i}\hat{s}\cos\theta_\alpha + \mathrm{i}\hat{n}\sin\theta_\alpha + \mathrm{i}N^2\cot^2\theta_\alpha}. \tag{3.69}$$

An independent ray calculation (details of which are given in the appendix) verifies that the terms common to (3.68) and (3.69) precisely match into the inner limit of the outer specularly reflected field. The extra term in (3.69) is therefore the inner form of the α-type acoustic head wave and this will provide the diffraction coefficient associated with elastic whispering gallery mode excitation and propagation at a fluid–solid interface. Details of this very involved calculation will be the subject of a subsequent paper.

4. Discussion

We have now completely specified the full acoustic response at points of elastic wave total internal reflection for the two separate cases of flat and curved fluid–solid interfaces. We placed most emphasis on the wavefields excited in the fluid, though of course the analysis that has been presented could be used to find that in the solid as well. Our justification for this prioritization is that it is the acoustic field which is more likely to be used for measurements in practical circumstances.

As has already been stated, these results now provide the launching coefficients for the radiative acoustic head waves and whispering gallery modes which form part of the outer scattered field. Not only that, but the derived inner diffraction structure considered here also provides the matching conditions to specify the transition solution required in the outer acoustic response to remove the singularity present in the far-field of the propagating head waves along the direction of critical *reflection*, i.e. along the direction of the specularly reflected ray associated with the critically incident acoustic ray. For the flat boundary case, this is manifested by the $\theta = \theta_\alpha$ singularity in (2.77) and a discussion of this case is offered in an appendix to a paper by Tew (1992b) and is taken no further here.

The techniques used here to solve this total internal reflection problem appear to be very robust and are amenable to some non-trivial extensions. One would be to consider more general incoming fields and another would be to analyse other boundary conditions, such as those at solid–solid interfaces.

In the former case, it might be possible to mimic beam incidence by locating the remote source considered in section 2 at a complex point, as in the construction due to Deschamps (1971). The expectation would then be that an analysis similar to that presented here would have to take place around a *complex* point of total internal reflection. This should present no methodological difficulties, though of course the implications for the outer ray picture must then be made in terms of complex ray fields (Chapman *et al* 1998). A similar situation would occur if the incoming field in section 3 were made evanescent (or inhomogeneous) by prescribing a complex angle of incidence. Indeed, the Stokes line structure embedded within the far-field asymptotics of the parabolic cylinder function describing the local acoustic fields would then be responsible for determining the propagation regions of such complex head waves and this in itself would be an interesting line of enquiry.

We have noted previously that the results of section 3 will be of use when looking at ‘whispering gallery’ mode excitation and propagation at convex fluid–solid boundaries.

This is currently under investigation for the cases of both open (i.e. infinite) and closed (i.e. finite) elastic solids. The latter case is of particular interest since if we consider the solid as an inclusion within an otherwise unbounded fluid, then in this high-frequency limit we can examine the global total acoustic response at sufficiently large distances for the inclusion to appear point-like. Under these circumstances, the incoming plane wave will dominate almost everywhere and the leading-order scattered field will appear as if it were emitted from a localized source, with a prescribed phase and an angular amplitude variation (or directivity). There will be a very narrow 'beam-like' region in which this is not the case and this region can be identified as the remains of the shadow zone at these large distances from the scatterer. The field here contains information about the obstructing elastic inclusion and it may well turn out to be the case that deductions about some of its material properties may be inferred from suitable measurements of this remaining shadow zone.

Of course, this cannot happen for an open elastic body, where a well-defined shadow zone persists at all distances. Even in this case, the likelihood is that acoustic head-wave propagation into shadow will be a dominant and significant feature, and the ability to construct its full structure using results similar to those presented here might be useful in practical aspects of acoustic microscopy or more general non-destructive testing techniques.

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Appendix

In this appendix, we consider in more detail some of the asymptotic matching calculations that are referred to in the main body of the paper. The results that we have presented so far are for the 'inner' diffraction solutions close to the points of total internal reflection and these solutions must match the 'outer' far-field solutions which we can calculate using ray theory.

Though many accounts of ray theory exist—see Keller and Lewis (1995) or for the case of coupled fluid–solid interactions, Tew (1992b), for example—we have chosen to include a brief account of the principal results here for completeness. We then follow this by details of the matching between certain components of the diffracted and the specularly reflected fields, as well as the ray construction of the outer form of the head-wave radiation.

Appendix A.1. Review of ray theory

For the acoustic response—exactly similar arguments follow for the elastic case (Tew, 1992b)—the far-field solution can be determined by considering the limit of the full Helmholtz equation

$$\left(\nabla^2 + k_0^2\right)\phi = 0 \qquad y > 0 \tag{A.1}$$

as $k_0 \to \infty$, subject to appropriate boundary conditions. This can be formalized by scaling $\boldsymbol{x} = L\bar{\boldsymbol{x}}$, $|\bar{\boldsymbol{x}}| = \mathcal{O}(1)$, and requiring that the normalized wavenumber $\kappa_0 = k_0 L$, be large.

We then assume that ϕ can be expressed in the form

$$\phi \sim e^{i\kappa_0 u(\bar{x},\bar{y})} \sum_{n=0}^{\infty} \frac{A_n(\bar{x},\bar{y})}{(i\kappa_0)^n} \tag{A.2}$$

identifying a phase $u(\bar{x},\bar{y})$ and an expansion for the amplitude in terms of reciprocal powers of κ_0. Substitution of (A.2) into (A.1) and extracting like powers of κ_0 yields the eikonal equation

$$\bar{\boldsymbol{\nabla}} u \cdot \bar{\boldsymbol{\nabla}} u = 1 \tag{A.3}$$

and the transport equations

$$A_0 \bar{\nabla}^2 u + 2\bar{\boldsymbol{\nabla}} A_0 \cdot \bar{\boldsymbol{\nabla}} u = 0 \tag{A.4}$$

$$A_n \bar{\nabla}^2 u + 2\bar{\boldsymbol{\nabla}} A_n \cdot \bar{\boldsymbol{\nabla}} u + \bar{\nabla}^2 A_{n-1} = 0 \qquad n = 1, 2, 3, \ldots. \tag{A.5}$$

The eikonal equation (A.3) can be solved parametrically in terms of arc-length $\bar{\tau}$ along the associated characteristics (or 'rays') by noting that along these rays, defined by

$$\frac{d\bar{\boldsymbol{x}}}{d\bar{\tau}} = \boldsymbol{p} \tag{A.6}$$

where $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y})$ and $\boldsymbol{p} = \bar{\boldsymbol{\nabla}} u$, we see that the eikonal equation is equivalent to

$$\frac{du}{d\bar{\tau}} = 1. \tag{A.7}$$

Hence, if $u = u_0(\bar{\rho})$ is given in terms of arc-length $\bar{\rho}$ on the scattering boundary $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}_0(\bar{\rho})$, then

$$u(\bar{\rho}, \bar{\tau}) = u_0(\bar{\rho}) + \bar{\tau} \tag{A.8}$$

now follows. Notice that we draw a distinction between arclength $\bar{\rho}$ in this calculation and arclength s as used in the main body of the paper; this is because the ray coordinates $(\bar{\rho}, \bar{\tau})$ are not the same as the boundary coordinates (s, n), though it is true that on the boundary (and only then), when $\bar{\tau} = 0 = n$, we have that $\bar{\rho} = s$ (in unscaled terms).

The definition $\boldsymbol{p} = \bar{\boldsymbol{\nabla}} u$ and (A.7) now imply that $\boldsymbol{p}$ is a constant vector, $\boldsymbol{p}_0(\bar{\rho})$ say, along the rays and so (A.6) can now be integrated to give the ray equations in the form

$$\bar{\boldsymbol{x}}(\bar{\tau}, \bar{\rho}) = \boldsymbol{p}_0(\bar{\rho})\bar{\tau} + \bar{\boldsymbol{x}}_0(\bar{\rho}). \tag{A.9}$$

Hence the ray directions are known once $\boldsymbol{p}_0(\bar{\rho})$ has been calculated, and this can be done by observing that

$$u_0'(\bar{\rho}) = \boldsymbol{p}_0(\bar{\rho}) \cdot \bar{\boldsymbol{x}}_0'(\bar{\rho}) \tag{A.10}$$

and

$$\boldsymbol{p}_0(\bar{\rho}) \cdot \boldsymbol{p}_0(\bar{\rho}) = 1 \tag{A.11}$$

equation (A.11) simply being a restatement of (A.3).

Along these rays, the leading-order transport equation (A.4) becomes the first-order ordinary differential equation

$$2\frac{dA_0}{d\bar{\tau}} + A_0 \bar{\nabla}^2 u = 0 \tag{A.12}$$

and a tedious calculation shows that the solution to this equation is

$$A_0(\bar{\rho}, \bar{\tau}) = A_0(\bar{\rho}, 0) \left\{ \frac{q_0(\bar{\rho})\, \bar{x}_0'(\bar{\rho}) - p_0(\bar{\rho})\, \bar{y}_0'(\bar{\rho})}{\bar{\tau}(q_0(\bar{\rho})\, p_0'(\bar{\rho}) - p_0(\bar{\rho})\, q_0'(\bar{\rho})) + q_0(\bar{\rho})\, \bar{x}_0'(\bar{\rho}) - p_0(\bar{\rho})\, \bar{y}_0'(\bar{\rho})} \right\}^{1/2} \tag{A.13}$$

where $\boldsymbol{p}_0(\bar{\rho}) = (p_0(\bar{\rho}), q_0(\bar{\rho}))$ and $A_0(\bar{\rho}, 0)$ is the amplitude on the scattering boundary $\bar{\tau} = 0$.

These results now completely specify the leading-order solution, since we have the phase u (A.8) and the amplitude A_0 (A.13) along the known rays (A.9).

As an example of this, we construct, using ray methods, the expression given in (2.77) for the acoustic head wave generated in the fluid by the total internal reflection of the α-type elastic wave in the solid.

In this case, the parametrization of the boundary $y = 0$ can be expressed in the form $\bar{\boldsymbol{x}}_0(\bar{\rho}) = (\bar{\rho}, 0)$. Also, the phase along the boundary of the totally internally reflected elastic field is, for $\alpha = p, s, \kappa_\alpha x$, where $\kappa_\alpha = k_\alpha L = \omega L/c_\alpha$, then this must also be the boundary evaluation of the phase of the associated acoustic field (which amounts to Snell's law being satisfied, essentially). This can be used to show that in the notation of the previous derivation,

$$u_0(\bar{\rho}) = \frac{c_0}{c_\alpha}\bar{\rho} = \cos\theta_\alpha \bar{\rho}. \tag{A.14}$$

In this case, equations (A.10) and (A.11) now give that

$$\frac{c_0}{c_\alpha} = p_0(\bar{\rho}) \tag{A.15}$$

and

$$1 = p_0^2(\bar{\rho}) + q_0^2(\bar{\rho}) \tag{A.16}$$

where we have again introduced the standard notation $\boldsymbol{p}_0(\bar{\rho}) = (p_0(\bar{\rho}), q_0(\bar{\rho}))$. In terms of the angle $\theta_\alpha = \cos^{-1}(c_0/c_\alpha)$, we are able to solve (A.15) and (A.16) to give

$$p_0(\bar{\rho}) = \cos\theta_\alpha \qquad q_0(\bar{\rho}) = \sin\theta_\alpha. \tag{A.17}$$

We are now in a position to state that the equations of the rays follow from (A.9) as

$$\bar{\boldsymbol{x}}(\bar{\tau}, \bar{\rho}) = \bar{\tau}(\cos\theta_\alpha, \sin\theta_\alpha) + (\bar{\rho}, 0) \tag{A.18}$$

along which (A.8) provides the phase as

$$u(\bar{\rho}, \bar{\tau}) = \bar{\rho}\cos\theta_\alpha + \bar{\tau}. \tag{A.19}$$

In fact, on taking components of (A.18), we are able to express u in (A.19) in terms of $\bar{x}$ and $\bar{y}$, the result being

$$u = \bar{x}\cos\theta_\alpha + \bar{y}\sin\theta_\alpha. \tag{A.20}$$

To calculate the leading-order amplitude variation, we can either substitute the results we have derived into (A.13) or we could go direct to (A.4), noting that (A.20) implies that $\bar{\nabla}^2 u = 0$. Hence, $\bar{\boldsymbol{\nabla}} A_0 \cdot \bar{\boldsymbol{\nabla}} u = 0$ and so, from (A.20) once more,

$$A_0(\bar{x}, \bar{y}) = f(\bar{x} - \bar{y}\cot\theta_\alpha) \tag{A.21}$$

where f is a function which we must determine.

At this point, we need further input to the calculation from the structure of the totally internally reflected elastic field. To be more precise, we need information about its amplitude variation with $\bar{x}$ along the boundary, which for the line source case considered in section 2 is well known to be proportioned to $(\bar{x} - \bar{h}\cot\theta_\alpha)^{-3/2}$ where $\bar{h} = h/L$ (see Tew 1992a for details). This implies that $f(\xi) \propto (\xi - \bar{h}\cot\theta_\alpha)^{-3/2}$, otherwise the boundary conditions will never be satisfied.

If we piece together the information we have so far derived, the upshot is that the leading-order ray ansatz now gives the acoustic head-wave radiation in the form

$$\phi^{(\alpha H)} \sim \mathrm{e}^{\mathrm{i}\kappa_0(\bar{x}\cos\theta_\alpha+\bar{y}\sin\theta_\alpha)} \frac{D_\alpha}{[\bar{x}-(\bar{y}+\bar{h})\cot\theta_\alpha]^{3/2}} \tag{A.22}$$

where D_α is the constant of proportionality that arises in the definition of f. Inverting the scalings in $\bar{x}$ and $\bar{y}$ and introducing plane polar coordinates (R,θ) centred on the image source point $(0,-\bar{h})$ allows (A.22) to be recast into the form

$$\phi^{(\alpha H)} \sim \frac{Q_\alpha \mathrm{e}^{\mathrm{i}k_0 R\cos(\theta_\alpha-\theta)}}{(k_0 R\sin(\theta_\alpha-\theta))^{3/2}} \qquad \theta_\alpha > \theta \tag{A.23}$$

for constant Q_α, which is identical to (2.77).

Appendix A.2. Matching between the 'inner' diffraction and 'outer' ray solutions

Our intention in this section is to provide further details of the asymptotic matching calculations that were performed to connect the inner and outer solutions around the points of total internal reflection. We shall do so for both the flat and curved boundary cases, beginning with the former.

Flat boundary case

Our intention here is to present the analysis that confirms one of the statements that follow equations (2.75) and (2.76). We begin by noting that if we work with the usual elastic displacement potentials ψ and χ and if we write the *total* acoustic potential Φ in the form

$$\Phi(x,y) = -\tfrac{1}{4}\mathrm{i}H_0^{(1)}(k_0R_0) - \tfrac{1}{4}\mathrm{i}H_0^{(1)}(k_0R) + \phi(x,y) \tag{A.24}$$

where $H_0^{(1)}$ is the usual Hankel function of the first kind and

$$R_0 = \sqrt{x^2+(y-h)^2} \qquad \text{and} \qquad R = \sqrt{x^2+(y+h)^2} \tag{A.25}$$

are the distances of a point (x,y) from the source and its image, respectively, then the boundary value problem is given by

$$\left(\nabla^2+k_0{}^2\right)\phi = 0 \qquad y>0 \tag{A.26}$$

$$\left(\nabla^2+k_P^2\right)\psi = 0 \qquad y<0 \tag{A.27}$$

$$\left(\nabla^2+k_S^2\right)\chi = 0 \qquad y<0 \tag{A.28}$$

with boundary conditions (all to be evaluated on $y=0$)

$$2\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\chi}{\partial y^2} - \frac{\partial^2\chi}{\partial x^2} = 0 \tag{A.29}$$

$$c_P^2\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right) - 2c_S^2\frac{\partial^2\psi}{\partial x^2} - 2c_S^2\frac{\partial^2\chi}{\partial x\partial y} + \frac{\varepsilon\mathrm{i}\omega c_P}{c_0}\phi = -\frac{\varepsilon\omega c_P}{2c_0}H_0^{(1)}(k_0R) \tag{A.30}$$

$$-\mathrm{i}\omega\left(\frac{\partial\psi}{\partial y} - \frac{\partial\chi}{\partial x}\right) = \frac{\partial\phi}{\partial y}. \tag{A.31}$$

Since we take the source to be remote from the boundary ($h \gg k_0{}^{-1}$), this justifies replacing the Hankel function by its leading-order asymptotic expansion such that the boundary condition (A.30) becomes

$$c_P^2\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)-2c_S^2\frac{\partial^2\psi}{\partial x^2}-2c_S^2\frac{\partial^2\chi}{\partial x\partial y}+\frac{\varepsilon\mathrm{i}\omega c_P}{c_0}\phi\sim-\frac{\varepsilon\omega c_P}{c_0}\sqrt{\frac{1}{2\pi k_0 R}}\mathrm{e}^{\mathrm{i}k_0R-\mathrm{i}\pi/4} \tag{A.32}$$

on $y=0$. To construct the specularly reflected field we scale $(x, y) = L(\bar{x}, \bar{y})$ as in section A.1, where $k_0L \gg 1$, and then the boundary value problem assumes the scaled form

$$(\bar{\nabla}^2+\kappa_0^2)\bar{\phi}=0 \qquad \bar{y}>0 \tag{A.33}$$

$$(\bar{\nabla}^2+\kappa_P^2)\bar{\psi}=0 \qquad \bar{y}<0 \tag{A.34}$$

$$(\bar{\nabla}^2+\kappa_S^2)\bar{\chi}=0 \qquad \bar{y}<0 \tag{A.35}$$

$$2\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}}+\frac{\partial^2\bar{\chi}}{\partial\bar{y}^2}-\frac{\partial^2\bar{\chi}}{\partial\bar{x}^2}=0 \qquad \bar{y}=0 \tag{A.36}$$

$$c_P^2\left(\frac{\partial^2\bar{\psi}}{\partial\bar{x}^2}+\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\right)-2c_S^2\frac{\partial^2\bar{\psi}}{\partial\bar{x}^2}-2c_S^2\frac{\partial^2\bar{\chi}}{\partial\bar{x}\partial\bar{y}}+\frac{\varepsilon\mathrm{i}\kappa_0{}^2c_P}{k_0}\bar{\phi}\sim-\frac{\kappa_0{}^2c_P}{k_0}\frac{\mathrm{e}^{\mathrm{i}\kappa_0\bar{R}-\mathrm{i}\pi/4}}{\sqrt{2\pi\kappa_0\bar{R}}} \qquad \bar{y}=0 \tag{A.37}$$

$$-\mathrm{i}\omega\left(\frac{\partial\bar{\psi}}{\partial\bar{y}}-\frac{\partial\bar{\chi}}{\partial\bar{x}}\right)=\frac{\partial\bar{\phi}}{\partial\bar{y}} \qquad \bar{y}=0 \tag{A.38}$$

where

$$(\phi,\psi,\chi)=\varepsilon(\bar{\phi},\bar{\psi},\bar{\chi}) \tag{A.39}$$

$\kappa_0 = k_0L$ (and suitably for κ_P and κ_S) and $R = L\bar{R}$. We consider the acoustic ray ansatz

$$\bar{\phi}(\bar{x},\bar{y})\sim\sum_{m=0}^{\infty}\frac{A_m^{(0)}(\bar{x},\bar{y})}{(\mathrm{i}\kappa_0)^m}\mathrm{e}^{\mathrm{i}\kappa_0u^{(0)}(\bar{x},\bar{y})} \tag{A.40}$$

with equivalent expansions for $\bar{\chi}$ and $\bar{\psi}$. Substitution of the acoustic and elastic ray ansatz into the boundary conditions (A.36)–(A.38) yields

$$\kappa_0u^{(0)}(\bar{x},0)=\kappa_Pu^{(P)}(\bar{x},0)=\kappa_Su^{(S)}(\bar{x},0)=\kappa_0\bar{R}(\bar{x},0) \tag{A.41}$$

where $u^{(P)}$ and $u^{(S)}$ are the phases of the P-type ($\bar{\psi}$) and S-type ($\bar{\chi}$) elastic ray fields, respectively.

In the notation of the previous section, (A.41) yields the initial data

$$u_0^{(\alpha)}(\bar{\rho})=\frac{c_\alpha}{c_0}\bar{R}(\bar{\rho},0)=\frac{c_\alpha}{c_0}\sqrt{\bar{\rho}^2+\bar{h}^2} \qquad \alpha=0,P,S \tag{A.42}$$

which, in turn, generates

$$p_0^{(0)}(\bar{\rho})=\frac{\bar{\rho}}{\sqrt{\bar{\rho}^2+\bar{h}^2}}=\cos\theta \qquad q_0^{(0)}(\bar{\rho})=\sin\theta \tag{A.43}$$

$$p_0^{(\alpha)}(\bar{\rho})=\frac{c_\alpha}{c_0}\frac{\bar{\rho}}{\sqrt{\bar{\rho}^2+\bar{h}^2}}=\frac{c_\alpha}{c_0}\cos\theta \qquad \alpha=P,S \tag{A.44}$$

and

$$q_0^{(\alpha)}(\bar{\rho})=\begin{cases}-\left(1-\dfrac{c_\alpha^2}{c_0^2}\cos^2\theta\right)^{1/2} & \theta>\theta_\alpha\\[2ex] -\mathrm{i}\left(\dfrac{c_\alpha^2}{c_0^2}\cos^2\theta-1\right)^{1/2} & \theta<\theta_\alpha\end{cases} \tag{A.45}$$

follow, where as usual $\theta_\alpha = \cos^{-1}(c_0/c_\alpha)$ and θ can be identified as a polar angle centred on the image source point $(0, -h)$. The choice of sign on $q_0^{(\alpha)}$ for $\alpha = 0, P, S$ ensures the radiation condition is satisfied. When $\theta < \theta_\alpha$, $\alpha = P, S$ the transmitted elastic α-rays are complex which gives rise to exponential decay in the solid.

We may now present the equations for the reflected rays in the fluid in the form

$$\bar{x} = \bar{\tau}\cos\theta + \bar{\rho} \qquad \bar{y} = \bar{\tau}\sin\theta \tag{A.46}$$

along which

$$u^{(0)}(\bar{\rho}, \bar{\tau}) = \bar{\tau} + \sqrt{\bar{\rho}^2 + \bar{h}^2}. \tag{A.47}$$

In the terms of the cylindrical coordinates $(\bar{R}, \theta)$ centred on the image source, we can write

$$\bar{x} = \bar{R}\cos\theta \qquad \bar{y} = \bar{R}\sin\theta - \bar{h} \tag{A.48}$$

and we can see, either geometrically or by direct calculation, that

$$u^{(0)}(\bar{x}, \bar{y}) = \bar{R}(\bar{x}, \bar{y}). \tag{A.49}$$

To calculate the amplitude variation, it is easiest to put this expression for $u^{(0)}$ in terms of $\bar{R}$ and $\bar{\theta}$ into (A.4) direct to obtain

$$2\frac{\partial A_0}{\partial \bar{R}} + \frac{A_0}{\bar{R}} = 0 \tag{A.50}$$

which has the general solution

$$A_0^{(0)}(\bar{R}, \theta) = \frac{F(\theta)}{\bar{R}^{1/2}}. \tag{A.51}$$

To calculate the directivity function $F(\theta)$, we can note that the leading-order boundary conditions may be conveniently expressed in matrix form

$$\begin{pmatrix} 0 & -\dfrac{2p_0^{(P)}q_0^{(P)}}{c_P^2} & \dfrac{1-2q_0^{(S)2}}{c_S^2} \\ -\dfrac{\mathrm{i}c_P\varepsilon}{c_0\omega} & 1-\dfrac{2c_S^2 p_0^{(S)2}}{c_P^2} & -2p_0^{(S)}q_0^{(S)} \\ -\dfrac{\mathrm{i}q_0^{(0)}}{c_0} & \dfrac{\omega q_0^{(P)}}{c_P} & -\dfrac{\omega p_0^{(S)}}{c_S} \end{pmatrix} \begin{pmatrix} A_0^{(0)} \\ \left(\dfrac{c_P}{c_0}\right)^{1/2} A_0^{(P)} \\ \left(\dfrac{c_S}{c_0}\right)^{1/2} A_0^{(S)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \dfrac{c_P}{\omega c_0}\dfrac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi\bar{R}}} \\ 0 \end{pmatrix} \qquad \bar{y} = 0 \tag{A.52}$$

from which

$$F(\theta) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}\gamma_{P0}}{\sqrt{2\pi}\left\{\sin\theta\left(\left(2\left(c_S^2/c_0^2\right)\cos^2\theta - 1\right)^2 + \left(4c_S^3/c_0^2c_P\right)\cos^2\theta\,\gamma_{P0}\gamma_{S0}\right) + \varepsilon\gamma_{P0}\right\}} \tag{A.53}$$

where $\gamma_{\alpha 0} = \left(1 - \left(c_\alpha^2/c_0^2\right)\cos^2\theta\right)^{1/2}$, $\alpha = P, S$ can be read off directly.

Therefore the leading-order solution is given by

$$\bar{\phi} \sim \frac{\mathrm{e}^{\mathrm{i}\kappa_0\bar{R}+\mathrm{i}\pi/4}\gamma_{P0}}{\sqrt{2\pi\kappa_0\bar{R}}\left\{\sin\theta(\bar{\rho})\left(\left(2\left(c_S^2/c_0^2\right)\cos^2\theta(\bar{\rho}) - 1\right)^2 + \left(4c_S^3/c_0^2c_P\right)\cos^2\theta(\bar{\rho})\gamma_{P0}\gamma_{S0}\right) + \varepsilon\gamma_{P0}\right\}}. \tag{A.54}$$

To relate these results to those of section 2, we note that the ray structure of the head wave (2.77) has been confirmed in section A.1. To finalize the calculation in section 2, we therefore need to check that the outer form of the reflected field, which we can calculate by adding to (A.54) the equivalent expressions representing the radiation from the actual and image sources, matches the inner form as presented in equations (2.73) and (2.74).

To do this, we introduce the inner scalings into this outer analysis and calculate that

$$k_0 R \sim \frac{k_0 h}{\sin\theta_\alpha} + \hat{x}\cos\theta_\alpha + \hat{y}\sin\theta_\alpha + \eta_\alpha^2 \tag{A.55}$$

$$\sqrt{k_0 R} \sim \sqrt{2}\eta_\alpha \tag{A.56}$$

$$(\theta - \theta_\alpha)^{1/2} \sim \frac{\mathrm{i}\sqrt{2}|\eta_\alpha|^{1/2}\delta_\alpha^{1/4}}{\sin^{1/2}\theta_\alpha} \tag{A.57}$$

with η_α given by (A.55), from which it follows that the limiting behaviour of the outer reflected field as it approaches the inner diffraction region is

$$\phi^{\mathrm{ref}} \sim \varepsilon\left(V_\alpha\delta_\alpha^{1/2} + H_\alpha|\eta_\alpha|^{1/2}\delta_\alpha^{3/4} + \frac{\mathrm{i}}{16|\eta_\alpha|^{3/2}}\right)\mathrm{e}^{(\mathrm{i}k_0 h/\sin\theta_\alpha)+\mathrm{i}\hat{x}\cos\theta_\alpha+\mathrm{i}\hat{y}\sin\theta_\alpha+\mathrm{i}\eta_\alpha^2} \tag{A.58}$$

in precise agreement with (2.73) and (2.74).

Curved boundary case

We now turn out attention to the ray and asymptotic matching calculations that confirm equations (3.68) and (3.69) for the case of plane wave insonification of a convex fluid–solid interface. In order to achieve this, we must first construct the leading-order ray solution for the geometrically reflected field in the fluid.

We begin by subtracting the incident field $\mathrm{e}^{\mathrm{i}kx}$ from the total acoustic potential ϕ, leaving the reflected field ϕ^{ref} to calculate. This results in the Helmholtz equations (A.26)–(A.28) for the acoustic and elastic potential functions, though these must now be expressed in (s, n) coordinates as in section 3, along with the boundary conditions (3.2)–(3.4).

We proceed by scaling the field equations and boundary conditions using $(s, n) = L(\bar{s}, \bar{n})$ and apply a ray ansatz (A.2) for ϕ^{ref}, with similar expansions for the elastic potential functions. If the eikonal phase functions associated with ψ and χ are $u^{(P)}$ and $u^{(S)}$, respectively, then the scaled versions of (3.2)–(3.4) immediately gives the eikonal boundary conditions

$$\kappa_0 u(\bar{\rho}, 0) = \kappa_P u^{(P)}(\bar{\rho},\ 0) = \kappa_S u^{(S)}(\bar{\rho}, 0) = \kappa_0 \bar{x}_0(\bar{\rho}). \tag{A.59}$$

Notice that we have switched from $\bar{s}$ to $\bar{\rho}$, as we are entitled to do on the boundary (but only there), in order to pose the problem in appropriate ray coordinates.

Equation (A.59) yields that

$$u(\bar{\rho}, 0) = u_0(\bar{\rho}) = \bar{x}_0(\bar{\rho}). \tag{A.60}$$

Furthermore, we can differentiate (A.60) to obtain the boundary condition

$$x_0'(\bar{\rho}) = \boldsymbol{t}(\bar{\rho}) \cdot \boldsymbol{p}_0(\bar{\rho}) \tag{A.61}$$

where $\boldsymbol{t}(\bar{\rho}) = \boldsymbol{x}_0'(\bar{\rho})$ is the local tangent vector. Since $\boldsymbol{p}_0(\bar{\rho}) \equiv \bar{\boldsymbol{\nabla}} u$, (A.61) implies that we can write down the boundary derivatives $\partial u/\partial\bar{\rho}$ and $\partial u/\partial\bar{n}$ on $\bar{n} = 0$ in the form

$$\frac{\partial u}{\partial\bar{\rho}} = -\cos\theta_i(\bar{\rho}) \qquad \frac{\partial u}{\partial n} = \sin\theta_i(\bar{\rho}) \tag{A.62}$$

where $\theta_i(\bar{\rho})$ is the angle of curvature of the boundary defined by

$$\boldsymbol{t}(\bar{\rho}) = (-\cos\theta_i(\bar{\rho}), -\sin\theta_i(\bar{\rho})) \tag{A.63}$$

and is depicted in figure 2. Notice that the first of the two expressions in (A.62) follows directly from (A.61) whilst the second is obtained by substituting this result into the eikonal equation (A.3).

An exactly similar procedure can be used to obtain the boundary derivatives of $u^{(P)}$ and $u^{(S)}$, and we need all of these functions because when we substitute the ray ansatz for all the potential functions into the boundary conditions (3.2)–(3.4), the highest-order terms correspond to differentiating the exponential pre-multipliers in the ray expansions and this necessarily introduces these various boundary derivatives.

If we denote the leading-order amplitudes of the reflected acoustic, transmitted longitudinal (P) and transmitted shear (S) type waves be A_0^{ref}, $A_0^{(P)}$ and $A_0^{(S)}$, respectively, then the procedure just described leads to the boundary conditions

$$\begin{pmatrix} 0 & \dfrac{-2c_S^2}{c_0c_P}\cos\theta_i(\bar{\rho})\gamma_P & 1-\dfrac{2c_S^2}{c_0^2}\cos^2\theta_i(\bar{\rho}) \\ \dfrac{\mathrm{i}\varepsilon c_P}{\omega c_0} & \dfrac{2c_S^2}{c_0^2}\cos^2\theta_i(\bar{\rho})-1 & -2\dfrac{c_S}{c_0}\cos\theta_i(\bar{\rho})\gamma_S \\ \mathrm{i}\sin\theta_i(\bar{\rho}) & \dfrac{c_0}{c_P}\omega\gamma_P & \omega\cos\theta_i(\bar{\rho}) \end{pmatrix} \begin{pmatrix} A_0^{\text{ref}}(\bar{\rho},0) \\ A_0^{(P)}(\bar{\rho},0) \\ A_0^{(S)}(\bar{\rho},0) \end{pmatrix} = \begin{pmatrix} 0 \\ -\dfrac{\mathrm{i}\varepsilon c_P}{\omega c_0} \\ \mathrm{i}\sin\theta_i(\bar{\rho}) \end{pmatrix} \tag{A.64}$$

where now

$$\gamma_\alpha = \left(1-\frac{c_\alpha^2}{c_0^2}\cos^2\theta_i(\bar{\rho})\right)^{1/2} \qquad \alpha = P, S. \tag{A.65}$$

Though we could use (A.64) to calculate each of the three boundary amplitudes, we concentrate here on the acoustic response, for which inversion of (A.64) yields

$$A^{\text{ref}}(\bar{\rho},0) = \frac{\sin\theta_i(\bar{\rho})\left\{\left(\left(2c_S^2/c_0^2\right)\cos^2\theta_i(\bar{\rho})-1\right)^2+\left(4c_S^3/c_0c_P\right)\cos^2\theta_i(\bar{\rho})\gamma_P\gamma_S\right\}-\varepsilon\gamma_P}{\sin\theta_i(\bar{\rho})\left\{\left(\left(2c_S^2/c_0^2\right)\cos^2\theta_i(\bar{\rho})-1\right)^2+\left(4c_S^3/c_0c_P\right)\cos^2\theta_i(\bar{\rho})\gamma_P\gamma_S\right\}+\varepsilon\gamma_P}. \tag{A.66}$$

To complete the determination of the reflected field, we must calculate the phase u and the amplitude A_0^{ref} away from the boundary. In fact, the phase u follows straightforwardly from our previous calculations as

$$u = \bar{x}_0(\bar{\rho}) + \bar{\tau} \tag{A.67}$$

and a simple calculation using (A.62) quickly demonstrates that the equations of the specularly reflected rays are

$$\bar{x} = \bar{x}_0(\bar{\rho}) + \bar{\tau}\cos 2\theta_i(\bar{\rho}) \qquad y = \bar{y}(\bar{\rho}) + \bar{\tau}\sin 2\theta_i(\bar{\rho}) \tag{A.68}$$

$$p_0(\bar{\rho}) = \cos 2\theta_i(\bar{\rho}) \qquad q_0(\bar{\rho}) = \sin 2\theta_i(\bar{\rho}). \tag{A.69}$$

If we now take these results and substitute them into (A.13) to find the amplitude, then

$$\phi^{\mathrm{ref}}(\bar{\rho}, \bar{\tau}) = A_0^{\mathrm{ref}}(\bar{\rho}, 0)\left(\frac{\sin\theta_i(\bar{\rho})}{\sin\theta_i(\bar{\rho}) + 2\kappa(\bar{\rho})\bar{\tau}}\right)^{1/2} \mathrm{e}^{\mathrm{i}\kappa_0(\bar{x}_0(\bar{\rho})+\bar{\tau})} \tag{A.70}$$

where $A_0^{\mathrm{ref}}(\bar{\rho}, 0)$ is given by (A.66), results.

This concludes the construction of the geometrically reflected acoustic field well away from the point of total internal reflection.

Returning to the inner diffraction analysis of section 3 in the main body of the paper, we see that that analysis required the scalings in (3.5). If we introduce those scalings into (A.70), making the necessary conversion from $(\bar{\rho}, \bar{\tau})$ to $(\hat{s}, \hat{n})$ coordinates, then we can compute the limiting behaviour of (A.70) in these inner coordinates. When we do this, we reproduce the terms representing the reflected field in (3.68) and (3.69) precisely, and the matching procedure is complete.

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